Math 304–504

Linear Algebra

Lecture 14:
The Vandermonde determinant.
Basis of a vector space.

Spanning set

Let S be a subset of a vector space V.

Definition. The **span** of the set S is the smallest subspace $W \subset V$ that contains S.

We say that the set S **spans** the subspace W or that S is a **spanning set** for W.

Theorem Let S be a nonempty subset of a vector space V. Then the span of S is the set of all vectors of the form

$$\boxed{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,}$$

where $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$ and $r_1, \dots, r_k \in \mathbb{R}$.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \ldots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

An infinite set $S \subset V$ is **linearly dependent** if there are some linearly dependent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in S$. Otherwise S is **linearly independent**.

Theorem The following conditions are equivalent:
(i) vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent;
(ii) one of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a linear combination of the other $k-1$ vectors.

Proposition If $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then

$$\mathrm{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k) = \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k).$$

Corollary Any spanning set for a vector space is minimal (i.e., cannot be reduced to a smaller spanning set) if and only if it is linearly independent.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$. The vector equation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0}$ is equivalent to the matrix equation $A\mathbf{x} = \mathbf{0}$, where

$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \qquad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

That is, A is the $n \times k$ matrix such that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are consecutive columns of A.

Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ *are linearly dependent if the matrix equation* $A\mathbf{x} = \mathbf{0}$ *has nonzero solutions.*

Theorem Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly dependent whenever $k > n$.

Problem Show that functions e^x, e^{2x}, and e^{3x} are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $ae^x + be^{2x} + ce^{3x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Differentiate this identity twice:

$$ae^x + 2be^{2x} + 3ce^{3x} = 0,$$

$$ae^x + 4be^{2x} + 9ce^{3x} = 0.$$

It follows that $A\mathbf{v} = \mathbf{0}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} ae^x \\ be^{2x} \\ ce^{3x} \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} ae^{x} \\ be^{2x} \\ ce^{3x} \end{pmatrix}.$$

To compute $\det A$, subtract the 1st row from the 2nd and the 3rd rows:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2.$$

Since A is invertible, we obtain

$$A\mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies ae^{x} = be^{2x} = ce^{3x} = 0$$
$$\implies a = b = c = 0$$

The Vandermonde determinant

Definition. The **Vandermonde determinant** is a determinant of the following form:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix},$$

where $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Examples.

- $\begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$

- $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ x_1 & x_2 - x_1 & x_3 \\ x_1^2 & x_2^2 - x_1^2 & x_3^2 \end{vmatrix} =$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x_1 & x_2 - x_1 & x_3 - x_1 \\ x_1^2 & x_2^2 - x_1^2 & x_3^2 - x_1^2 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ x_2^2 - x_1^2 & x_3^2 - x_1^2 \end{vmatrix} =$$

$$= (x_2 - x_1)(x_3^2 - x_1^2) - (x_3 - x_1)(x_2^2 - x_1^2) =$$

$$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$$

Theorem

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Corollary The Vandermonde determinant is not equal to 0 if and only if the numbers $x_1, x_2, \ldots, x_n$ are distinct.

Let $x_1, x_2, \ldots, x_n$ be distinct real numbers.

Theorem For any $b_1, b_2, \ldots, b_n \in \mathbb{R}$ there exists a unique polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ of degree less than n such that $p(x_i) = b_i$, $1 \le i \le n$.

$$\begin{cases} a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1} = b_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_{n-1} x_2^{n-1} = b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_{n-1} x_n^{n-1} = b_n \end{cases}$$

$a_0, a_1, \ldots, a_{n-1}$ are unknowns. The coefficient matrix is the transpose of a Vandermonde matrix.

Theorem Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct real numbers. Then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x}$ are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $r_1 e^{\lambda_1 x} + r_2 e^{\lambda_2 x} + \cdots + r_n e^{\lambda_n x} = 0$ for all $x \in \mathbb{R}$, where $r_1, r_2, \ldots, r_n$ are costants.

Differentiate this identity $n-1$ times:

$$\lambda_1 r_1 e^{\lambda_1 x} + \lambda_2 r_2 e^{\lambda_2 x} + \cdots + \lambda_n r_n e^{\lambda_n x} = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$\lambda_1^{n-1} r_1 e^{\lambda_1 x} + \lambda_2^{n-1} r_2 e^{\lambda_2 x} + \cdots + \lambda_n^{n-1} r_n e^{\lambda_n x} = 0.$$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} \begin{pmatrix} r_1 e^{\lambda_1 x} \\ r_2 e^{\lambda_2 x} \\ \vdots \\ r_n e^{\lambda_n x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Longrightarrow \quad r_1 = r_2 = \cdots = r_n = 0$$

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Suppose that a set $S \subset V$ is a basis for V.
"Spanning set" means that any vector $\mathbf{v} \in V$ can be represented as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \ldots, r_k \in \mathbb{R}$. "Linearly independent" implies that the above representation is unique:

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = r_1'\mathbf{v}_1 + r_2'\mathbf{v}_2 + \cdots + r_k'\mathbf{v}_k$$
$$\implies (r_1 - r_1')\mathbf{v}_1 + (r_2 - r_2')\mathbf{v}_2 + \cdots + (r_k - r_k')\mathbf{v}_k = \mathbf{0}$$
$$\implies r_1 - r_1' = r_2 - r_2' = \ldots = r_n - r_n' = 0$$

Examples. • Standard basis for $\mathbb{R}^n$:
$\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0)$, ...,
$\mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$.
Indeed, $(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

• Polynomials $1, x, x^2, \dots, x^{n-1}$ form a basis for
$\mathcal{P}_n = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

Theorem If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a spanning set for a vector space V and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\} \subset V$ is a linearly independent set, then $m \geq n$.

Corollary 1 If $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ are both bases for a vector space V, then $m = n$.

Corollary 2 Every basis for $\mathbb{R}^n$ consists of n vectors.

Corollary 3 Given n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$, the following conditions are equivalent:
(i) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

Problem Show that vectors $\mathbf{v}_1 = (1, -1, 1, -1)$, $\mathbf{v}_2 = (1, 0, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4, 8)$ form a basis for $\mathbb{R}^4$.

It is enough to show that the vectors are linearly independent. To do this, we need to check invertibility of the 4×4 matrix whose columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{vmatrix} = - \begin{vmatrix} -1 & 1 & 2 \\ 1 & 1 & 4 \\ -1 & 1 & 8 \end{vmatrix} = - \begin{vmatrix} 0 & 1 & 2 \\ 2 & 1 & 4 \\ 0 & 1 & 8 \end{vmatrix} =$$

$$= -(-2) \begin{vmatrix} 1 & 2 \\ 1 & 8 \end{vmatrix} = 2 \cdot 6 = 12 \neq 0.$$