Math 304–504

Linear Algebra

Lecture 15:
Basis and dimension.

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Equivalently, a subset $S \subset V$ is a basis for V if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \ldots, r_k \in \mathbb{R}$.

Examples. • Standard basis for $\mathbb{R}^n$:
$\mathbf{e}_1 = (1, 0, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0, 0)$,$\ldots$,
$\mathbf{e}_n = (0, 0, 0, \ldots, 0, 1)$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

• Polynomials $1, x, x^2, \ldots, x^{n-1}$ form a basis for
$\mathcal{P}_n = \{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} : a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

Dimension

Theorem Any vector space V has a basis. All bases for V are of the same cardinality.

Definition. The **dimension** of a vector space V, denoted $\dim V$, is the cardinality of its bases.

Remark. By definition, two sets are of the same cardinality if there exists a one-to-one correspondence between their elements.

For a finite set, the cardinality is the number of its elements.

For an infinite set, the cardinality is a more sophisticated notion. For example, $\mathbb{Z}$ and $\mathbb{R}$ are infinite sets of different cardinalities while $\mathbb{Z}$ and $\mathbb{Q}$ are infinite sets of the same cardinality.

Examples. • $\dim \mathbb{R}^n = n$

• $\mathcal{M}_{2,2}(\mathbb{R})$: the space of 2×2 matrices
$\dim \mathcal{M}_{2,2}(\mathbb{R}) = 4$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m\times n$ matrices
$\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

• $\mathcal{P}_n$: polynomials of degree less than n
$\dim \mathcal{P}_n = n$

• $\mathcal{P}$: the space of all polynomials
$\dim \mathcal{P} = \infty$

• $\{\mathbf{0}\}$: the trivial vector space
$\dim \{\mathbf{0}\} = 0$

Problem. Find the dimension of the plane $x + 2z = 0$ in $\mathbb{R}^3$.

The general solution of the equation $x + 2z = 0$ is

$$\begin{cases} x = -2s \\ y = t \\ z = s \end{cases} \qquad (t, s \in \mathbb{R})$$

That is, $(x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)$.

Hence the plane is the span of vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$. These vectors are linearly independent as they are not parallel.

Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis so that the dimension of the plane is 2.

Bases for $\mathbb{R}^n$

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$.

Theorem 1 If $m < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ do not span $\mathbb{R}^n$.

Theorem 2 If $m > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are linearly dependent.

Theorem 3 If $m = n$ then the following conditions are equivalent:
(i) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (as they are not parallel), but they do not span $\mathbb{R}^3$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-2) = 2 \neq 0.$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^3$ (because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ already span $\mathbb{R}^3$), but they are linearly dependent.

How to find a basis?

Theorem Let S be a subset of a vector space V. Then the following conditions are equivalent:

(i) S is a linearly independent spanning set for V, i.e., a basis;

(ii) S is a minimal spanning set for V;

(iii) S is a maximal linearly independent subset of V.

"Minimal spanning set" means "remove any element from this set, and it is no longer a spanning set".

"Maximal linearly independent subset" means "add any element of V to this set, and it will become linearly dependent".

Theorem Let V be a vector space. Then
(i) any spanning set for V can be reduced to a minimal spanning set;
(ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

Equivalently, any spanning set contains a basis, while any linearly independent set is contained in a basis.

Corollary A vector space is finite-dimensional if and only if it is spanned by a finite set.

How to find a basis?

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis.

Proposition Let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ be a spanning set for a vector space V. If $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then $\mathbf{v}_1, \dots, \mathbf{v}_k$ is also a spanning set for V.

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k =$$
$$= (t_0r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0r_k + t_k)\mathbf{v}_k.$$

How to find a basis?

Approach 2. Build a maximal linearly independent set adding one vector at a time.

If the vector space V is trivial, it has the empty basis.

If $V \neq \{\mathbf{0}\}$, pick any vector $\mathbf{v}_1 \neq \mathbf{0}$.
If $\mathbf{v}_1$ spans V, it is a basis. Otherwise pick any vector $\mathbf{v}_2 \in V$ that is not in the span of $\mathbf{v}_1$.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ span V, they constitute a basis. Otherwise pick any vector $\mathbf{v}_3 \in V$ that is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

And so on...

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1,1,0)$, $\mathbf{w}_2 = (0,1,1)$, $\mathbf{w}_3 = (2,3,1)$, and $\mathbf{w}_4 = (1,1,1)$.

To pare this spanning set, we need to find a relation of the form $r_1\mathbf{w}_1+r_2\mathbf{w}_2+r_3\mathbf{w}_3+r_4\mathbf{w}_4 = \mathbf{0}$, where $r_i \in \mathbb{R}$ are not all equal to zero. Equivalently,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this system of linear equations for r_1, r_2, r_3, r_4, we apply row reduction.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(reduced row echelon form)}$$

$$\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \iff \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}$$

General solution: $(r_1, r_2, r_3, r_4) = (-2t, -t, t, 0)$, $t \in \mathbb{R}$.

Particular solution: $(r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$.

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

We have obtained that $2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3 = \mathbf{0}$.
Hence any of vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ can be dropped.
For instance, $V = \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4)$.

Let us check whether vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4$ are linearly independent:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

They are!!! It follows that $V = \mathbb{R}^3$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ is a basis for V.