Math 304–504

Linear Algebra

Lecture 16:
Basis and coordinates.

Basis and dimension

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Theorem Any vector space V has a basis. If V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition. The **dimension** of a vector space V, denoted $\dim V$, is the number of elements in any of its bases.

Examples. • $\dim \mathbb{R}^n = n$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices
$\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

• $\mathcal{P}_n$: polynomials of degree less than n
$\dim \mathcal{P}_n = n$

• $\mathcal{P}$: the space of all polynomials
$\dim \mathcal{P} = \infty$

• $\{\mathbf{0}\}$: the trivial vector space
$\dim \{\mathbf{0}\} = 0$

Theorem Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in $\mathbb{R}^n$. Then the following conditions are equivalent:

(i) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

Theorem Let S be a subset of a vector space V. Then the following conditions are equivalent:

(i) S is a linearly independent spanning set for V, i.e., a basis;
(ii) S is a minimal spanning set for V;
(iii) S is a maximal linearly independent subset of V.

How to find a basis?

Theorem Let V be a vector space. Then
(i) any spanning set for V can be reduced to a minimal spanning set;
(ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

That is, any spanning set contains a basis, while any linearly independent set is contained in a basis.

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis.

Approach 2. Build a maximal linearly independent set adding one vector at a time.

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for $\mathbb{R}^3$.

Hint 1. $\mathbf{v}_1$ and $\mathbf{v}_2$ span the plane $x + 2z = 0$.

The vector $\mathbf{v}_3 = (1, 1, 1)$ does not lie in the plane $x + 2z = 0$, hence it is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

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Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Hint 2. At least one of vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ is a desired one.

Let us check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ are two bases for $\mathbb{R}^3$:

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \qquad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \;\mapsto\; \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

is a one-to-one correspondence between V and $\mathbb{R}^n$. This correspondence respects linear operations in V and in $\mathbb{R}^n$.

Vectors $\mathbf{u}_1=(2,1)$ and $\mathbf{u}_2=(3,1)$ form a basis for $\mathbb{R}^2$.

Problem 1. Find coordinates of the vector $\mathbf{v}=(7,4)$ with respect to the basis $\mathbf{u}_1,\mathbf{u}_2$.

The desired coordinates x,y satisfy

$$\mathbf{v}=x\mathbf{u}_1+y\mathbf{u}_2 \iff \begin{cases} 2x+3y=7 \\ x+y=4 \end{cases} \iff \begin{cases} x=5 \\ y=-1 \end{cases}$$

Problem 2. Find the vector $\mathbf{w}$ whose coordinates with respect to the basis $\mathbf{u}_1,\mathbf{u}_2$ are $(7,4)$.

$$\mathbf{w}=7\mathbf{u}_1+4\mathbf{u}_2=7(2,1)+4(3,1)=(26,11)$$

Given a vector $\mathbf{v} \in \mathbb{R}^2$, let (x, y) be its standard coordinates, i.e., coordinates with respect to the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ and let (x', y') be its coordinates with respect to the basis $\mathbf{u}_1 = (2, 1)$, $\mathbf{u}_2 = (3, 1)$.

Problem. Find a relation between (x, y) and (x', y').

By definition, $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{u}_1 + y'\mathbf{u}_2$.
In standard coordinates,

$$\begin{pmatrix} x \\ y \end{pmatrix} = x' \begin{pmatrix} 2 \\ 1 \end{pmatrix} + y' \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Change of coordinates in $\mathbb{R}^n$

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis for $\mathbb{R}^n$.

Take any vector $\mathbf{x} \in \mathbb{R}^n$. Let $(x_1, x_2, \ldots, x_n)$ be the standard coordinates of $\mathbf{x}$ and $(x'_1, x'_2, \ldots, x'_n)$ be the coordinates of $\mathbf{x}$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Problem 1. Given the standard coordinates $(x_1, x_2, \ldots, x_n)$, find the nonstandard coordinates $(x'_1, x'_2, \ldots, x'_n)$.

Problem 2. Given the nonstandard coordinates $(x'_1, x'_2, \ldots, x'_n)$, find the standard coordinates $(x_1, x_2, \ldots, x_n)$.

Let $\mathbf{u}_j = (u_{1j}, u_{2j}, \dots, u_{nj})$, $j = 1, 2, \dots, n$.
Then

$$\mathbf{x} = x_1'\mathbf{u}_1 + x_2'\mathbf{u}_2 + \cdots + x_n'\mathbf{u}_n$$

$$\begin{aligned} &= x_1'(u_{11}\mathbf{e}_1 + u_{21}\mathbf{e}_2 + \cdots + u_{n1}\mathbf{e}_n) \\ &+ x_2'(u_{12}\mathbf{e}_1 + u_{22}\mathbf{e}_2 + \cdots + u_{n2}\mathbf{e}_n) \\ &+ \cdots\cdots\cdots \\ &+ x_n'(u_{1n}\mathbf{e}_1 + u_{2n}\mathbf{e}_2 + \cdots + u_{nn}\mathbf{e}_n) \end{aligned}$$

$$\begin{aligned} &= (u_{11}x_1' + u_{12}x_2' + \cdots + u_{1n}x_n')\mathbf{e}_1 \\ &+ (u_{21}x_1' + u_{22}x_2' + \cdots + u_{2n}x_n')\mathbf{e}_2 \\ &+ \cdots\cdots\cdots \\ &+ (u_{n1}x_1' + u_{n2}x_2' + \cdots + u_{nn}x_n')\mathbf{e}_n. \end{aligned}$$

At the same time, $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$.

It follows that

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}.$$

The matrix $U = (u_{ij})$ does not depend on the vector $\mathbf{x}$.

Columns of U are coordinates of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ with respect to the standard basis.

U is called the **transition matrix** from the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$.

This solves Problem 2. To solve Problem 1, we have to use the inverse matrix U^{-1}, which is the transition matrix from $\mathbf{e}_1, \dots, \mathbf{e}_n$ to $\mathbf{u}_1, \dots, \mathbf{u}_n$.

Problem. Find the transition matrix from the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in $\mathbb{R}^3$ to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

The transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$U = \left(\begin{array}{c|c|c} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right).$$

The transition matrix from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is the inverse matrix $A = U^{-1}$.

The inverse matrix can be computed using row reduction.

$$(U \mid I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Subtract the 1st row from the 2nd row:

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Subtract the 2nd row from the 3rd row:

$$\rightarrow \left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 1 & 1 & 0 & 0 \\ 0 & \color{red}{1} & 0 & -1 & 1 & 0 \\ 0 & 0 & \color{red}{1} & 1 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

Subtract the 3rd row from the 1st row:

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) = (I \mid U^{-1})$$

Thus $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The columns of A are coordinates of vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Change of coordinates: general case

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be two bases for a vector space V. Take any vector $\mathbf{x} \in V$. Let $(x_1, x_2, \dots, x_n)$ be the coordinates of $\mathbf{x}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and $(x'_1, x'_2, \dots, x'_n)$ be the coordinates of $\mathbf{x}$ with respect to $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. Then

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} = \begin{pmatrix} u_{11} & \dots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \dots & u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

where $U = (u_{ij})$ is the **transition matrix** from the basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \dots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \dots, \mathbf{u}_n$.

Problem. Find the transition matrix from the basis $p_1(x) = 1$, $p_2(x) = x + 1$, $p_3(x) = (x + 1)^2$ to the basis $q_1(x) = 1$, $q_2(x) = x$, $q_3(x) = x^2$ for the vector space $\mathcal{P}_3$.

We have to find coordinates of the polynomials p_1, p_2, p_3 with respect to the basis q_1, q_2, q_3:

$p_1(x) = 1 = q_1(x)$,
$p_2(x) = x + 1 = q_1(x) + q_2(x)$,
$p_3(x) = (x+1)^2 = x^2+2x+1 = q_1(x)+2q_2(x)+q_3(x)$.

Thus the transition matrix is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.