Math 304–504

Linear Algebra

Lecture 17:
Change of coordinates (continued).
Rank and nullity of a matrix

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \;\mapsto\; \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

is a one-to-one correspondence between V and $\mathbb{R}^n$. This correspondence respects linear operations in V and in $\mathbb{R}^n$.

Change of coordinates

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be two bases for a vector space V. Take any vector $\mathbf{x} \in V$. Let $(x_1, x_2, \ldots, x_n)$ be the coordinates of $\mathbf{x}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and $(x'_1, x'_2, \ldots, x'_n)$ be the coordinates of $\mathbf{x}$ with respect to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Then

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} = \begin{pmatrix} u_{11} & \ldots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \ldots & u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

where $U = (u_{ij})$ is the **transition matrix** from the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

Example. Vectors $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, and $\mathbf{u}_3 = (1, 1, 1)$ form a basis for $\mathbb{R}^3$.

The transition matrix from the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The transition matrix from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Problem. Find the transition matrix from the basis $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (1, 2, 1)$ to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

To change coordinates from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, we first change coordinates from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and then from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$U_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}.$$

Transition matrix from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$:

$$U_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$:

$$U_2 U_1 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

Nullspace

Let $A = (a_{ij})$ be an $m \times n$ matrix.

Definition. The **nullspace** of the matrix A, denoted $N(A)$, is the set of all n-dimensional column vectors $\mathbf{x}$ such that $\boxed{A\mathbf{x} = \mathbf{0}.}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The nullspace $N(A)$ is the solution set of a system of linear homogeneous equations (with A as the coefficient matrix).

Let $A = (a_{ij})$ be an $m \times n$ matrix.

Theorem The nullspace $N(A)$ is a subspace of the vector space $\mathbb{R}^n$.

Proof: We have to show that $N(A)$ is nonempty, closed under addition, and closed under scaling.

First of all, $A\mathbf{0} = \mathbf{0} \implies \mathbf{0} \in N(A) \implies N(A)$ is not empty.

Secondly, if $\mathbf{x}, \mathbf{y} \in N(A)$, i.e., if $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$, then
$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0} \implies \mathbf{x}+\mathbf{y} \in N(A)$.

Thirdly, if $\mathbf{x} \in N(A)$, i.e., if $A\mathbf{x} = \mathbf{0}$, then for any $r \in \mathbb{R}$ one has $A(r\mathbf{x}) = r(A\mathbf{x}) = r\mathbf{0} = \mathbf{0} \implies r\mathbf{x} \in N(A)$.

Definition. The dimension of the nullspace $N(A)$ is called the **nullity** of the matrix A.

Problem. Find the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Elementary row operations do not change the nullspace.
Let us convert A to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{cases}$$

General element of $N(A)$:

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (t + 2s, -2t - 3s, t, s) \\ &= t(1, -2, 1, 0) + s(2, -3, 0, 1), \quad t, s \in \mathbb{R}. \end{aligned}$$

Vectors $(1, -2, 1, 0)$ and $(2, -3, 0, 1)$ forms a basis for $N(A)$.
Thus the nullity of the matrix A is 2.

Row space

Definition. The **row space** of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by rows of A.

The dimension of the row space is called the **rank** of the matrix A.

Theorem 1 Elementary row operations do not change the row space of a matrix.

Theorem 2 If a matrix A is in row echelon form, then the nonzero rows of A are linearly independent.

Corollary The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

Theorem 3 The rank of a matrix A plus the nullity of A equals the number of columns of A.

Problem. Find the rank of the matrix

$$A = \begin{pmatrix} -1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix}.$$

Elementary row operations do not change the row space. Let us convert A to row echelon form:

$$\begin{pmatrix} -1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 1 & 0 & 1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Vectors $(1, 0, 1, -2)$ and $(0, 0, 0, 1)$ form a basis for the row space of A. Thus the rank of A is 2.

Remark. The rank of A equals the number of nonzero rows in the row echelon form, which equals the number of leading entries.

The nullity of A equals the number of free variables in the corresponding system, which equals the number of columns without leading entries.

Consequently, rank+nullity is the number of all columns in the matrix A.

Theorem 1 Elementary row operations do not change the row space of a matrix.

Proof: Suppose that A and B are $m \times n$ matrices such that B is obtained from A by an elementary row operation. Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be the rows of A and $\mathbf{b}_1, \ldots, \mathbf{b}_m$ be the rows of B. We have to show that $\mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_m)$.

Observe that any row $\mathbf{b}_i$ of B belongs to $\mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$. Indeed, either $\mathbf{b}_i = \mathbf{a}_j$ for some $1 \leq j \leq m$, or $\mathbf{b}_i = r\mathbf{a}_i$ for some scalar $r \neq 0$, or $\mathbf{b}_i = \mathbf{a}_i + r\mathbf{a}_j$ for some $j \neq i$ and $r \in \mathbb{R}$.

It follows that $\mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_m) \subset \mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$.

Now the matrix A can also be obtained from B by an elementary row operation. By the above,

$$\mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m) \subset \mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_m).$$

Problem. Find the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Alternative solution: Clearly, the rows of A are linearly independent. Therefore the rank of A is 2. Since

$$(\text{rank of } A) + (\text{nullity of } A) = 4,$$

it follows that the nullity of A is 2.