Math 304–504

Linear Algebra

Lecture 18:
Column space of a matrix.
Linear transformations.
Kernel and range.

Nullspace

Let $A = (a_{ij})$ be an $m \times n$ matrix.

Definition. The **nullspace** of the matrix A, denoted $N(A)$, is the set of all n-dimensional column vectors $\mathbf{x}$ such that $\boxed{A\mathbf{x} = \mathbf{0}.}$

Theorem 1 The nullspace $N(A)$ is a subspace of the vector space $\mathbb{R}^n$.

Theorem 2 Elementary row operations do not change the nullspace of a matrix.

Definition. The dimension of the nullspace $N(A)$ is called the **nullity** of the matrix A.

Row space

Definition. The **row space** of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by rows of A.

The dimension of the row space is called the **rank** of the matrix A.

Theorem 1 Elementary row operations do not change the row space of a matrix.

Theorem 2 If a matrix A is in row echelon form, then the nonzero rows of A are linearly independent.

Corollary The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

Theorem 3 The rank of a matrix A plus the nullity of A equals the number of columns of A.

Column space

Definition. The **column space** of an $m \times n$ matrix A is the subspace of $\mathbb{R}^m$ spanned by columns of A.

Theorem 1 The column space of a matrix A coincides with the row space of the transpose matrix A^T.

Theorem 2 Elementary column operations do not change the column space of a matrix.

Theorem 3 Elementary row operations do not change the dimension of the column space of a matrix (although they can change the column space).

Theorem 4 For any matrix, the row space and the column space have the same dimension.

Problem. Find a basis for the column space of the matrix

$$A = \begin{pmatrix} -1 & -1 & 0 & 2 \\ 1 & 1 & 0 & -1 \\ 2 & 2 & 0 & 0 \end{pmatrix}.$$

The column space of A coincides with the row space of A^T. To find a basis, we convert A^T to row echelon form:

$$\begin{pmatrix} -1 & 1 & 2 \\ -1 & 1 & 2 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 1 & 2 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus vectors $(1, -2, -1)$ and $(0, 1, 4)$ form a basis for the column space of the matrix A.

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

A linear mapping $\ell : V \to \mathbb{R}$ is called a **linear functional** on V.

If $V_1 = V_2$ (or if both V_1 and V_2 are functional spaces) then a linear mapping $L : V_1 \to V_2$ is called a **linear operator**.

Properties of linear mappings

Let $L : V_1 \to V_2$ be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$

for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.

$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2)$,
$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) =$
$= r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2) + r_3L(\mathbf{v}_3)$, and so on.

- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.

$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2$.

- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v})$.

Linear functionals

- $V = \mathbb{R}^n$, $\ell(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$.
- $V = C[a, b]$, $\ell(f) = f(a)$.
- $V = C^1[a, b]$, $\ell(f) = f'(b)$.
- $V = C[a, b]$, $\ell(f) = \displaystyle\int_a^b f(x)\,dx$.
- $V = C[a, b]$, $\ell(f) = \displaystyle\int_a^b g(x)f(x)\,dx$,

where $g \in C[a, b]$.

Linear operators

- $V = \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix and $\mathbf{x}$ is regarded as a column vector.
- $V = C[a, b]$, $L(f) = gf$, where $g \in C[a, b]$.
- $V_1 = C^1[a, b]$, $V_2 = C[a, b]$, $L(f) = f'$.
- $V = C[a, b]$, $(L(f))(x) = \int_a^x f(\xi)\, d\xi$.
- $V = C[a, b]$, $(L(f))(x) = \int_a^b G(x, \xi) f(\xi)\, d\xi$, where $G \in C([a, b] \times [a, b])$.

Linear differential operators

- an ordinary differential operator

$$L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on $\mathbb{R}$.
That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- Laplace's operator $\Delta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Range and kernel

Let V, W be vector spaces and $L : V \to W$ be a linear mapping.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L, denoted $\ker L$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem (i) The range of L is a subspace of W.
(ii) The kernel of L is a subspace of V.

Example. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The kernel $\ker L$ is the nullspace of the matrix.
The range $L(\mathbb{R}^3)$ is the column space of the matrix.

The range of L is spanned by vectors $(1, 1, 1)$, $(0, 2, 0)$, and $(-1, -1, -1)$. It follows that $L(\mathbb{R}^3)$ is the plane spanned by $(1, 1, 1)$ and $(0, 1, 0)$.

To find $\ker L$, we apply row reduction to the matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $(x, y, z) \in \ker L$ if $x - z = y = 0$.
It follows that $\ker L$ is the line spanned by $(1, 0, 1)$.