Math 304–504

Linear Algebra

Lecture 1:
Systems of linear equations.

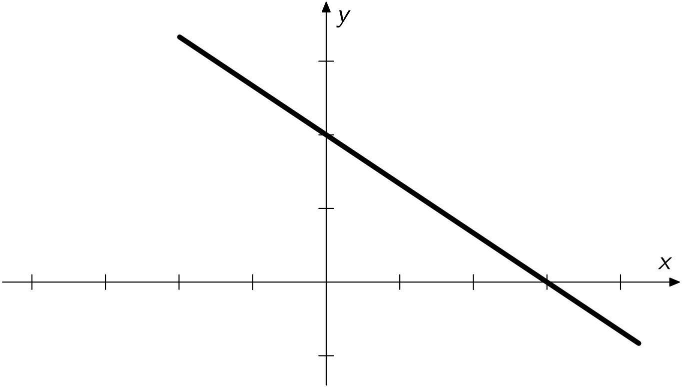
Linear equation

The equation $\boxed{2x + 3y = 6}$ is called *linear* because its solution set is a line in $\mathbb{R}^2$.

A *solution* of the equation is a pair of numbers $(\alpha, \beta) \in \mathbb{R}^2$ such that $2\alpha + 3\beta = 6$.

For example, $(3, 0)$ and $(0, 2)$ are solutions.

Alternatively, we can write the first solution as $x = 3$, $y = 0$.



$$2x + 3y = 6$$

General equation of a line: $\boxed{ax + by = c,}$
where x, y are variables and a, b, c are constants (except for the case $a = b = 0$).

Definition. A *linear equation* in variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,}$$

where $a_1, \ldots, a_n$, and b are constants.

A *solution* of the equation is an array of numbers $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{R}^n$ such that

$$a_1 \gamma_1 + a_2 \gamma_2 + \cdots + a_n \gamma_n = b.$$

System of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Here $x_1, x_2, \ldots, x_n$ are variables and a_{ij}, b_j are constants.

A *solution* of the system is a common solution of all equations in the system.

Plenty of problems in mathematics and real world require solving systems of linear equations.

Problem Find the point of intersection of the lines $x - y = -2$ and $2x + 3y = 6$ in $\mathbb{R}^2$.

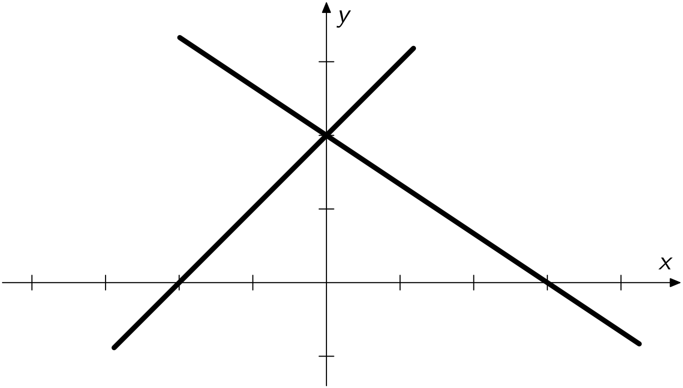
$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases} \iff \begin{cases} x = y - 2 \\ 2x + 3y = 6 \end{cases} \iff$$

$$\begin{cases} x = y - 2 \\ 2(y - 2) + 3y = 6 \end{cases} \iff \begin{cases} x = y - 2 \\ 5y = 10 \end{cases} \iff$$

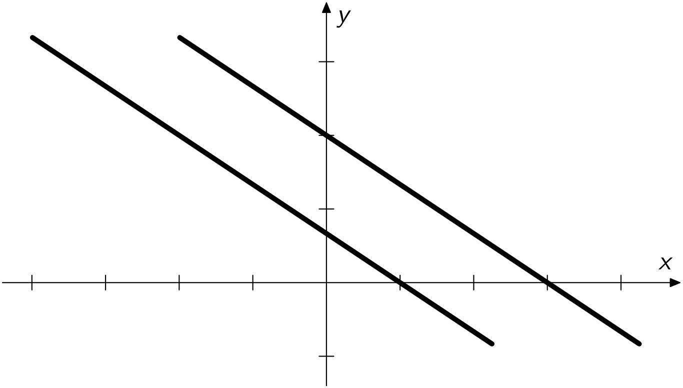
$$\begin{cases} x = y - 2 \\ y = 2 \end{cases} \iff \begin{cases} x = 0 \\ y = 2 \end{cases}$$

Solution: the lines intersect at the point $(0, 2)$.

Remark. The symbol of equivalence $\iff$ means that two systems have the same solutions.



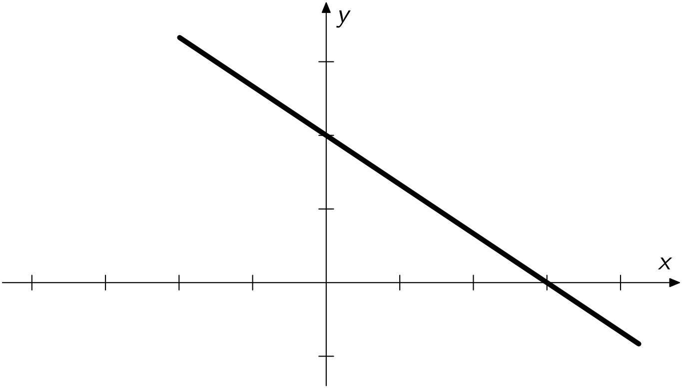
$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases} \qquad x = 0,\ y = 2$$



$$\begin{cases} 2x + 3y = 2 \\ 2x + 3y = 6 \end{cases}$$

inconsistent system

(no solutions)



$$\begin{cases} 4x + 6y = 12 \\ 2x + 3y = 6 \end{cases} \iff 2x + 3y = 6$$

Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

The algorithm reduces the number of variables (as well as the number of equations), hence it stops after a finite number of steps.

After the algorithm stops, the system is simplified so that it is clear how to solve it.

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Solve the 1st equation for x:

$$\begin{cases} x = y + 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

Eliminate x from the 2nd and 3rd equations:

$$\begin{cases} x = y + 2 \\ 2(y + 2) - y - z = 3 \\ (y + 2) + y + z = 6 \end{cases}$$

Simplify:

$$\begin{cases} x = y + 2 \\ y - z = -1 \\ 2y + z = 4 \end{cases}$$

Now the 2nd and 3rd equations form the system of two linear equations in two variables.

Solve the 2nd equation for y, then eliminate y from the 3rd equation:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 2y + z = 4 \end{cases} \qquad \begin{cases} x = y + 2 \\ y = z - 1 \\ 2(z - 1) + z = 4 \end{cases}$$

Simplify:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 3z = 6 \end{cases}$$

The elimination is completed. Now the system is easily solved by back substitution.

That is, we find z from the 3rd equation, then substitute it in the 2nd equation and find y, then substitute y and z in the 1st equation and find x.

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ z = 2 \end{cases} \qquad \begin{cases} x = y + 2 \\ y = 1 \\ z = 2 \end{cases} \qquad \begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases}$$

System of linear equations:

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Solution: $(x, y, z) = (3, 1, 2)$

Gaussian elimination

Gaussian elimination is a modification of the elimination method that uses only so-called *elementary operations*.

Elementary operations for systems of linear equations:

(1) to multiply an equation by a nonzero scalar;

(2) to add an equation multiplied by a scalar to another equation;

(3) to interchange two equations.

Proposition Any elementary operation can be undone by applying another elementary operation.

Theorem Applying elementary operations to a system of linear equations does not change the solution set of the system.

Proof: It is clear that after an elementary operation we do not lose any solution. Since the operation can be undone by another elementary operation, neither we get any garbage solutions.

Operation 1: multiply the ith equation by $r \neq 0$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\Longrightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots \\ (ra_{i1})x_1 + (ra_{i2})x_2 + \cdots + (ra_{in})x_n = rb_i \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

To undo the operation, multiply the ith equation by r^{-1}.

Operation 2: add r times the ith equation to the jth equation.

$$\begin{cases} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \cdots\cdots\cdots\cdots \end{cases} \Longrightarrow$$

$$\begin{cases} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ (a_{j1} + ra_{i1})x_1 + \cdots + (a_{jn} + ra_{in})x_n = b_j + rb_i \\ \cdots\cdots\cdots\cdots \end{cases}$$

To undo the operation, add $-r$ times the ith equation to the jth equation.

Operation 3: interchange the ith and jth equations.

$$\begin{cases} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \cdots\cdots\cdots\cdots \end{cases}$$

$$\Longrightarrow \begin{cases} \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \end{cases}$$

To undo the operation, apply it once more.