Math 304–504
Linear Algebra

Lecture 21:
Matrix of a linear transformation.

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Basic properties of linear mappings:

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$ for all $k \geq 1$, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V_1$, and $r_1, \ldots, r_k \in \mathbb{R}$.
- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.
- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

Matrix transformations

Any $m \times n$ matrix A gives rise to a transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $L(\mathbf{x}) \in \mathbb{R}^m$ are regarded as column vectors.

This transformation is **linear**.

Example. $$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ be the standard basis for $\mathbb{R}^3$. Then vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$ are columns of the matrix.

Theorem 1 Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V. Then

(i) any linear mapping $L : V \to W$ is uniquely determined by vectors $L(\mathbf{v}_1), L(\mathbf{v}_2), \dots, L(\mathbf{v}_n)$;

(ii) for any vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \in W$ there exists a linear mapping $L : V \to W$ such that $L(\mathbf{v}_i) = \mathbf{w}_i$, $1 \leq i \leq n$.

Theorem 2 Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. The columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for $\mathbb{R}^n$.

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \;\mapsto\; \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

provides a one-to-one correspondence between V and $\mathbb{R}^n$. Besides, this mapping is linear.

Matrix of a linear mapping

Let V, W be vector spaces and $f : V \to W$ be a linear map. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis. Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ be a basis for W and $g_2 : W \to \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ g_1 \big\downarrow & & \big\downarrow g_2 \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$. It is represented as $\mathbf{v} \mapsto A\mathbf{v}$, where A is an $m \times n$ matrix.

A is called the **matrix of** f with respect to bases $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1, \dots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \dots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \dots, \mathbf{w}_m$.

Examples. • $D : \mathcal{P}_3 \to \mathcal{P}_2$, $(Dp)(x) = p'(x)$.

Let A_D be the matrix of D with respect to the bases $1, x, x^2$ and $1, x$. Columns of A_D are coordinates of polynomials $D1$, Dx, Dx^2 w.r.t. the basis $1, x$.

$$D1 = 0,\ Dx = 1,\ Dx^2 = 2x \implies A_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

• $L : \mathcal{P}_3 \to \mathcal{P}_3$, $(Lp)(x) = p(x+1)$.

Let A_L be the matrix of L w.r.t. the basis $1, x, x^2$.
$L1 = 1$, $Lx = 1 + x$, $Lx^2 = (x+1)^2 = 1 + 2x + x^2$.

$$\implies A_L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem. Consider a linear operator $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let N be the desired matrix. Columns of N are coordinates of the vectors $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ w.r.t. the basis $\mathbf{v}_1, \mathbf{v}_2$.

$$L(\mathbf{v}_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad L(\mathbf{v}_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Clearly, $L(\mathbf{v}_2) = \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2$.

$$L(\mathbf{v}_1) = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 \iff \begin{cases} 3\alpha + 2\beta = 4 \\ \alpha + \beta = 1 \end{cases} \iff \begin{cases} \alpha = 2 \\ \beta = -1 \end{cases}$$

Thus $N = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

Problem. Find the matrix of the identity mapping $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(\mathbf{x}) = \mathbf{x}$ with respect to the bases:

(i) $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ and $\mathbf{e}_1, \mathbf{e}_2$;
(ii) $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$ and $\mathbf{v}_1, \mathbf{v}_2$;
(iii) $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{e}_1, \mathbf{e}_2$;
(iv) $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{v}_1, \mathbf{v}_2$.

The desired matrix is the **transition matrix** from the first basis to the second one:

$$A_1 = A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$

$$A_4 = A_3^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}.$$

Change of basis

Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$. It is given by $L(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$, where A is an $n \times n$ matrix.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a nonstandard basis for $\mathbb{R}^n$. If we change the standard coordinates in $\mathbb{R}^n$ to the coordinates relative to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, then the operator L is given by $L(\mathbf{y}) = B\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^n$, where B is another $n \times n$ matrix.

A is the matrix of L relative to the standard basis.
B is the matrix of L relative to the basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

Columns of A are vectors $L(\mathbf{e}_1), \ldots, L(\mathbf{e}_n)$.
Columns of B are coordinates of vectors $L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n)$ relative to the basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

Problem 1. Given the matrix A, find the matrix B.

Problem 2. Given the matrix B, find the matrix A.

Let $\mathbf{u}_j = (u_{1j}, u_{2j}, \dots, u_{nj})$, $j = 1, 2, \dots, n$.
Consider the transition matrix $U = (u_{ij})$ from the basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ to the standard basis.

Theorem $A = UBU^{-1}$, $B = U^{-1}AU$.

Proof: It is enough to prove the 2nd formula as

$$B{=}U^{-1}AU \implies UBU^{-1}{=}U(U^{-1}AU)U^{-1}{=}(UU^{-1})A(UU^{-1}){=}A.$$

Take any $\mathbf{x} \in \mathbb{R}^n$ and let $\mathbf{y}$ be the (vector of) nonstandard coordinates of $\mathbf{x}$.

Then $U\mathbf{y}$ are standard coordinates of $\mathbf{x}$
$\implies$ $AU\mathbf{y}$ are standard coordinates of $L(\mathbf{x})$
$\implies$ $U^{-1}AU\mathbf{y}$ are nonstandard coordinates of $L(\mathbf{x})$.

Problem. Consider a linear operator $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let S be the matrix of L with respect to the standard basis, N be the matrix of L with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$, and U be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2$ to $\mathbf{e}_1, \mathbf{e}_2$. Then $N = U^{-1}SU$.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$

$$N = U^{-1}SU = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$