

Math 304–504

Linear Algebra

**Lecture 22:**  
**Similarity.**

## Basis and coordinates

If  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for a vector space  $V$ , then any vector  $\mathbf{v} \in V$  has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where  $x_i \in \mathbb{R}$ . The coefficients  $x_1, x_2, \dots, x_n$  are called the **coordinates** of  $\mathbf{v}$  with respect to the ordered basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .

The mapping

$$\text{vector } \mathbf{v} \mapsto \text{its coordinates } (x_1, x_2, \dots, x_n)$$

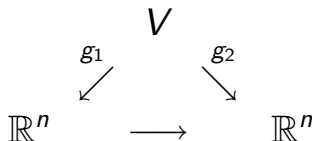
provides a one-to-one correspondence between  $V$  and  $\mathbb{R}^n$ . This mapping is linear.

## Change of coordinates

Let  $V$  be a vector space.

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be a basis for  $V$  and  $g_1 : V \rightarrow \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  be another basis for  $V$  and  $g_2 : V \rightarrow \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.



The composition  $g_2 \circ g_1^{-1}$  is a linear mapping of  $\mathbb{R}^n$  to itself. It is represented as  $\mathbf{v} \mapsto U\mathbf{v}$ , where  $U$  is an  $n \times n$  matrix.

$U$  is called the **transition matrix** from  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  to  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ . Columns of  $U$  are coordinates of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  with respect to the basis  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ .

## Matrix of a linear mapping

Let  $V, W$  be vector spaces and  $f : V \rightarrow W$  be a linear map.

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be a basis for  $V$  and  $g_1 : V \rightarrow \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.

Let  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$  be a basis for  $W$  and  $g_2 : W \rightarrow \mathbb{R}^m$  be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ g_1 \downarrow & & \downarrow g_2 \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

The composition  $g_2 \circ f \circ g_1^{-1}$  is a linear mapping of  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . It is represented as  $\mathbf{v} \mapsto A\mathbf{v}$ , where  $A$  is an  $m \times n$  matrix.

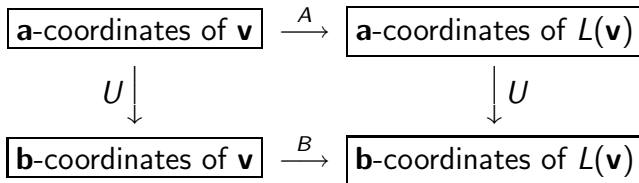
$A$  is called the **matrix of  $f$**  with respect to bases  $\mathbf{v}_1, \dots, \mathbf{v}_n$  and  $\mathbf{w}_1, \dots, \mathbf{w}_m$ . Columns of  $A$  are coordinates of vectors  $f(\mathbf{v}_1), \dots, f(\mathbf{v}_n)$  with respect to the basis  $\mathbf{w}_1, \dots, \mathbf{w}_m$ .

## Change of basis for a linear operator

Let  $L : V \rightarrow V$  be a linear operator on a vector space  $V$ .

Let  $A$  be the matrix of  $L$  relative to a basis  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  for  $V$ . Let  $B$  be the matrix of  $L$  relative to another basis  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$  for  $V$ .

Let  $U$  be the transition matrix from the basis  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  to  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ .



It follows that  $UA = BU$ .

Then  $A = U^{-1}BU$  and  $B = UAU^{-1}$ .

**Problem.** Consider a linear operator  $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of  $L$  with respect to the basis  $\mathbf{v}_1 = (3, 1)$ ,  $\mathbf{v}_2 = (2, 1)$ .

Let  $S$  be the matrix of  $L$  with respect to the standard basis,  $N$  be the matrix of  $L$  with respect to the basis  $\mathbf{v}_1, \mathbf{v}_2$ , and  $U$  be the transition matrix from  $\mathbf{v}_1, \mathbf{v}_2$  to  $\mathbf{e}_1, \mathbf{e}_2$ . Then  $N = U^{-1}SU$ .

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$

$$\begin{aligned} N = U^{-1}SU &= \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

**Problem.** Let  $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ .

Find the matrix of the linear operator  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $L(\mathbf{x}) = A\mathbf{x}$  with respect to the basis  $\mathbf{v}_1 = (-1, 1, 0)$ ,  $\mathbf{v}_2 = (1, 1, 0)$ ,  $\mathbf{v}_3 = (-1, 0, 1)$ .

Let  $B$  be the desired matrix. The columns of  $B$  are coordinates of the vectors  $L(\mathbf{v}_1)$ ,  $L(\mathbf{v}_2)$ ,  $L(\mathbf{v}_3)$  with respect to the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ .

$$L(\mathbf{v}_1) = (0, 0, 0), \quad L(\mathbf{v}_2) = (2, 2, 0) = 2\mathbf{v}_2, \\ L(\mathbf{v}_3) = (-2, 0, 2) = 2\mathbf{v}_3.$$

$$\text{Thus } B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

**Problem.** Let  $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ . Find  $A^{16}$ .

It follows from the solution of the previous problem that  $A = UBU^{-1}$ , where

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that  $A^2 = AA = UBU^{-1}UBU^{-1} = UB^2U^{-1}$ ,  
 $A^3 = A^2A = UB^2U^{-1}UBU^{-1} = UB^3U^{-1}$ , and so on.



In particular,  $A^{16} = UB^{16}U^{-1}$ .

Clearly,  $B^{16} = \text{diag}(0, 2^{16}, 2^{16}) = 2^{15}B$ .

Hence  $A^{16} = U(2^{15}B)U^{-1} = 2^{15}UBU^{-1} = 2^{15}A$   
 $= 32768 A$ .

$$A^{16} = \begin{pmatrix} 32768 & 32768 & -32768 \\ 32768 & 32768 & 32768 \\ 0 & 0 & 65536 \end{pmatrix}.$$

## Similarity

*Definition.* An  $n \times n$  matrix  $B$  is said to be **similar** to an  $n \times n$  matrix  $A$  if  $\boxed{B = S^{-1}AS}$  for some nonsingular  $n \times n$  matrix  $S$ .

*Remark.* Two  $n \times n$  matrices are similar if and only if they represent the same linear operator on  $\mathbb{R}^n$  with respect to some bases.

**Theorem** Similarity is an *equivalence relation*, i.e.,

- (i) any square matrix  $A$  is similar to itself;
- (ii) if  $B$  is similar to  $A$ , then  $A$  is similar to  $B$ ;
- (iii) if  $A$  is similar to  $B$  and  $B$  is similar to  $C$ , then  $A$  is similar to  $C$ .

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*Proof:* (i)  $A = I^{-1}AI$ .

(ii) If  $B = S^{-1}AS$  then  $A = SBS^{-1} = (S^{-1})^{-1}BS^{-1}$ .

(iii) If  $A = S^{-1}BS$  and  $B = T^{-1}CT$  then  
 $A = S^{-1}T^{-1}CTS = (TS)^{-1}C(TS)$ .

**Theorem** If  $A$  and  $B$  are similar matrices then they have the same (i) determinant, (ii) trace = the sum of diagonal entries, (iii) rank, and (iv) nullity.