Math 304–504

Linear Algebra

Lecture 22:
Similarity.

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \;\mapsto\; \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

provides a one-to-one correspondence between V and $\mathbb{R}^n$. This mapping is linear.

Change of coordinates

Let V be a vector space.
Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.
Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccccc} & & V & & \\ & {}^{g_1}\swarrow & & \searrow^{g_2} & \\ \mathbb{R}^n & & \longrightarrow & & \mathbb{R}^n \end{array}$$

The composition $g_2 \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to itself.
It is represented as $\mathbf{v} \mapsto U\mathbf{v}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Matrix of a linear mapping

Let V, W be vector spaces and $f : V \to W$ be a linear map. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis. Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ be a basis for W and $g_2 : W \to \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

$$
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
g_1 \Big\downarrow & & \Big\downarrow g_2 \\
\mathbb{R}^n & \longrightarrow & \mathbb{R}^m
\end{array}
$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$. It is represented as $\mathbf{v} \mapsto A\mathbf{v}$, where A is an $m \times n$ matrix.

A is called the **matrix of** f with respect to bases $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1, \dots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \dots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \dots, \mathbf{w}_m$.

Change of basis for a linear operator

Let $L : V \to V$ be a linear oprator on a vector space V.

Let A be the matrix of L relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ for V.

Let U be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$.

$$\begin{array}{ccc}
\boxed{\mathbf{a}\text{-coordinates of } \mathbf{v}} & \xrightarrow{A} & \boxed{\mathbf{a}\text{-coordinates of } L(\mathbf{v})} \\
U \big\downarrow & & \big\downarrow U \\
\boxed{\mathbf{b}\text{-coordinates of } \mathbf{v}} & \xrightarrow{B} & \boxed{\mathbf{b}\text{-coordinates of } L(\mathbf{v})}
\end{array}$$

It follows that $UA = BU$.

Then $A = U^{-1}BU$ and $B = UAU^{-1}$.

Problem. Consider a linear operator $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let S be the matrix of L with respect to the standard basis, N be the matrix of L with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$, and U be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2$ to $\mathbf{e}_1, \mathbf{e}_2$. Then $N = U^{-1}SU$.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$

$$N = U^{-1}SU = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Problem. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Find the matrix of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{x}) = A\mathbf{x}$ with respect to the basis $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (-1, 0, 1)$.

Let B be the desired matrix. The columns of B are coordinates of the vectors $L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$L(\mathbf{v}_1) = (0, 0, 0)$, $L(\mathbf{v}_2) = (2, 2, 0) = 2\mathbf{v}_2$,
$L(\mathbf{v}_3) = (-2, 0, 2) = 2\mathbf{v}_3$.

Thus $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Problem. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Find A^{16}.

It follows from the solution of the previous problem that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $A^2 = AA = UBU^{-1}UBU^{-1} = UB^2U^{-1}$, $A^3 = A^2A = UB^2U^{-1}UBU^{-1} = UB^3U^{-1}$, and so on.

In particular, $A^{16} = UB^{16}U^{-1}$.

Clearly, $B^{16} = \mathrm{diag}(0, 2^{16}, 2^{16}) = 2^{15}B$.

Hence $A^{16} = U(2^{15}B)U^{-1} = 2^{15}UBU^{-1} = 2^{15}A$
$= 32768\,A$.

$$A^{16} = \begin{pmatrix} 32768 & 32768 & -32768 \\ 32768 & 32768 & 32768 \\ 0 & 0 & 65536 \end{pmatrix}.$$

Similarity

Definition. An $n \times n$ matrix B is said to be **similar** to an $n \times n$ matrix A if $\boxed{B = S^{-1}AS}$ for some nonsingular $n \times n$ matrix S.

Remark. Two $n \times n$ matrices are similar if and only if they represent the same linear operator on $\mathbb{R}^n$ with respect to some bases.

Theorem Similarity is an *equivalence relation*, i.e.,
(i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

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Proof: **(i)** $A = I^{-1}AI$.
(ii) If $B = S^{-1}AS$ then $A = SBS^{-1} = (S^{-1})^{-1}BS^{-1}$.
(iii) If $A = S^{-1}BS$ and $B = T^{-1}CT$ then
$A = S^{-1}T^{-1}CTS = (TS)^{-1}C(TS)$.

Theorem If A and B are similar matrices then they have the same (i) determinant, (ii) trace = the sum of diagonal entries, (iii) rank, and (iv) nullity.