Math 304–504

Linear Algebra

Lecture 32:
Diagonalization (continued).
Complex eigenvalues and eigenvectors.

Diagonalization

Let L be a linear operator on a finite-dimensional vector space V. Then the following conditions are equivalent:

- the matrix of L with respect to some basis is diagonal;
- there exists a basis for V formed by eigenvectors of L.

The operator L is **diagonalizable** if it satisfies these conditions.

Let A be an $n \times n$ matrix. Then the following conditions are equivalent:

- A is the matrix of a diagonalizable operator;
- A is similar to a diagonal matrix, i.e., it is represented as $A = UBU^{-1}$, where the matrix B is diagonal;
- there exists a basis for $\mathbb{R}^n$ formed by eigenvectors of A.

The matrix A is **diagonalizable** if it satisfies these conditions. Otherwise A is called **defective**.

Problem. Diagonalize the matrix $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$.

We need to find a diagonal matrix B and an invertible matrix U such that $A = UBU^{-1}$.

Suppose that $\mathbf{v}_1 = (x_1, y_1)$, $\mathbf{v}_2 = (x_2, y_2)$ is a basis for $\mathbb{R}^2$ formed by eigenvectors of A, i.e., $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for some $\lambda_i \in \mathbb{R}$. Then we can take

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad U = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

Note that U is the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2$ to the standard basis.

Problem. Diagonalize the matrix $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$.

Characteristic equation of A: $\begin{vmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = 0$.

$(4-\lambda)(1-\lambda) = 0 \implies \lambda_1 = 4,\ \lambda_2 = 1.$

Associated eigenvectors: $\mathbf{v}_1 = (1,0),\ \mathbf{v}_2 = (-1,1)$.

Thus $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Problem. Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find A^5.

We know that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then $A^5 = UBU^{-1}UBU^{-1}UBU^{-1}UBU^{-1}UBU^{-1}$

$$= UB^5U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1024 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1024 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1024 & 1023 \\ 0 & 1 \end{pmatrix}.$$

Problem. Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find a matrix C such that $C^2 = A$.

We know that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Suppose that $D^2 = B$ for some matrix D. Let $C = UDU^{-1}$. Then $C^2 = UDU^{-1}UDU^{-1} = UD^2U^{-1} = UBU^{-1} = A$.

We can take $D = \begin{pmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

There are *two obstructions* to existence of a basis consisting of eigenvectors. They are illustrated by the following examples.

Example 1. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\det(A - \lambda I) = (\lambda - 1)^2$. Hence $\lambda = 1$ is the only eigenvalue. The associated eigenspace is the line $t(1, 0)$.

Example 2. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = \lambda^2 + 1$.

$\implies$ no real eigenvalues or eigenvectors

(However there are *complex* eigenvalues/eigenvectors.)

Evolution of numbers

Natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$.

$$x + 5 = 3, \quad x = ?$$

Integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

$$7x = 5, \quad x = ?$$

Rational numbers: $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, \ n \neq 0\}$.

$$x^2 = 2, \quad x = ?$$

Real numbers: $\mathbb{R}$.

$$x^2 + 1 = 0, \quad x = ?$$

Complex numbers

$\mathbb{C}$: complex numbers.

Complex number: $\boxed{z = x + iy,}$

where $x, y \in \mathbb{R}$ and $i^2 = -1$.

$i = \sqrt{-1}$: imaginary unit

Alternative notation: $z = x + yi$.

$x =$ real part of z,
$iy =$ imaginary part of z

$y = 0 \implies z = x$ (real number)
$x = 0 \implies z = iy$ (purely imaginary number)

We add and multiply complex numbers as polynomials in i (but keep in mind that $i^2 = -1$).

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Examples. • $(1 + i) - (3 + 5i) = (1 - 3) + (i - 5i)$ $= -2 - 4i$;

• $(1 + i)(3 + 5i) = 1 \cdot 3 + i \cdot 3 + 1 \cdot 5i + i \cdot 5i$ $= 3 + 3i + 5i + 5i^2 = 3 + 3i + 5i - 5 = -2 + 8i$;

• $2i(3 - 2i) = 6i - 4i^2 = 4 + 6i$;

• $(2 + 3i)(2 - 3i) = 4 - 9i^2 = 4 + 9 = 13$;

• $i^3 = -i, \;\; i^4 = 1, \;\; i^5 = i$.

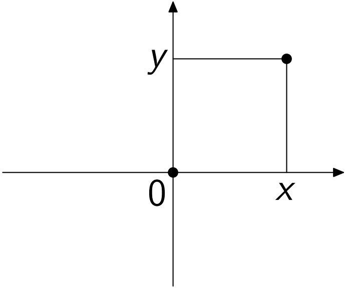
Let $z = \cos\alpha + i\sin\alpha$, $w = \cos\beta + i\sin\beta$, where $\alpha, \beta \in \mathbb{R}$.

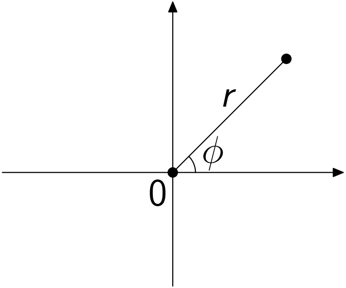
$zw = \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$.

By definition, $e^{i\phi} = \cos\phi + i\sin\phi$ for any $\phi \in \mathbb{R}$.
Then $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$, $\alpha, \beta \in \mathbb{R}$.

Geometric representation

Any complex number $z = x + iy$ is represented by the vector/point $(x, y) \in \mathbb{R}^2$.





$$x = r\cos\phi, \quad y = r\sin\phi$$

$$\implies z = r(\cos\phi + i\sin\phi) = re^{i\phi}.$$

$$\boxed{z = r(\cos\phi + i\sin\phi) = re^{i\phi}}$$

$r \geq 0$ is the **modulus** of z (denoted $|z|$).
$|x + iy| = \sqrt{x^2 + y^2}$.

$\phi \in \mathbb{R}$ is the **argument** of z (determined up to adding a multiple of 2π).

$$z_1 = r_1 e^{i\phi_1}, \;\; z_2 = r_2 e^{i\phi_2} \quad \Longrightarrow \quad z_1 z_2 = r_1 r_2 e^{i(\phi_1 + \phi_2)}$$

Division

If $z = re^{i\phi}$, then $z^{-1} = r^{-1}e^{-i\phi}$ because $re^{i\phi} \cdot r^{-1}e^{-i\phi} = rr^{-1}e^{i(\phi-\phi)} = e^{i0} = 1.$

$$z_1 = r_1 e^{i\phi_1}, \ \ z_2 = r_2 e^{i\phi_2} \quad \Longrightarrow \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)}$$

Given $z = x + iy$, the **complex conjugate** of z is $\bar{z} = x - iy$. The conjugacy $z \mapsto \bar{z}$ is the reflection of $\mathbb{C}$ in the real line.

$z\bar{z} = (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2.$

$$z^{-1} = \frac{\bar{z}}{|z|^2}, \qquad (x+iy)^{-1} = \frac{x - iy}{x^2 + y^2}.$$

Examples. • $i^{-1} = \frac{i}{i^2} = -i;$

• $\frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{1-i^2} = \frac{1}{2} + \frac{1}{2}i;$

• $\frac{2+3i}{1+2i} = \frac{(2+3i)(1-2i)}{(1+2i)(1-2i)} = \frac{8-i}{5} = 1.6 - 0.2i.$

Roots of unity

Problem. Solve the equation $z^n - 1 = 0$ $(n \geq 1)$.

Let $z = re^{i\phi}$ $(r > 0,\ \phi \in \mathbb{R})$. Then $z^n = r^n e^{in\phi}$.

Hence $z^n = 1$ if $r^n = 1$ and $n\phi = 2\pi k$, $k \in \mathbb{Z}$.

That is, $r = 1$, $\phi = 2\pi k/n$, $k \in \mathbb{Z}$.

Solutions: $z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, $0 \leq k \leq n-1$.

Cubic roots of unity: 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Roots of unity of degree 4: $1, i, -1, -i$.

Fundamental Theorem of Algebra
Any polynomial of degree $n \geq 1$, with complex coefficients, has exactly n roots (counting with multiplicities).

Equivalently, if

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$, then there exist complex numbers $z_1, z_2, \ldots, z_n$ such that

$$p(z) = a_n (z - z_1)(z - z_2) \ldots (z - z_n).$$

Complex eigenvalues/eigenvectors

Example. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\det(A - \lambda I) = \lambda^2 + 1$.

Characteristic roots: $\lambda_1 = i$ and $\lambda_2 = -i$.

Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$ and $\mathbf{v}_2 = (1, i)$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

$\mathbf{v}_1$, $\mathbf{v}_2$ is a basis of eigenvectors. *In which space?*

Complexification

Instead of the real vector space $\mathbb{R}^2$, we consider a complex vector space $\mathbb{C}^2$ (all complex numbers are admissible as scalars).

The linear operator $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(\mathbf{x}) = A\mathbf{x}$ is replaced by the complexified linear operator $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F(\mathbf{x}) = A\mathbf{x}$.

The vectors $\mathbf{v}_1 = (1, -i)$ and $\mathbf{v}_2 = (1, i)$ form a basis for $\mathbb{C}^2$.

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

$$A_\phi \mathbf{v}_1 = \begin{pmatrix} \cos\phi + i\sin\phi \\ -i\cos\phi + \sin\phi \end{pmatrix} = e^{i\phi}\mathbf{v}_1, \qquad A_\phi \mathbf{v}_2 = e^{-i\phi}\mathbf{v}_2.$$