Math 304–504
Linear Algebra

**Lecture 33:
Review for Test 2.**

# Topics for Test 2

*Linear transformations  (Leon 4.1–4.3)*
- Matrix transformations
- Matrix of a linear mapping
- Similar matrices

*Orthogonality  (Leon 5.1–5.6)*
- Inner products and norms
- Orthogonal complement
- Least squares problems
- The Gram-Schmidt orthogonalization process

*Eigenvalues and eigenvectors  (Leon 6.1, 6.3)*
- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Diagonalization

## Sample problems for Test 2

**Problem 1 (20 pts.)** Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the vector space of $2 \times 2$ matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Find the matrix of the operator $L$ with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Problem 2 (30 pts.)** Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

**(i)** Find an orthonormal basis for $V$.

**(ii)** Find an orthonormal basis for the orthogonal complement $V^\perp$.

**Problem 3 (30 pts.)** Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

**(i)** Find all eigenvalues of the matrix $A$.

**(ii)** For each eigenvalue of $A$, find an associated eigenvector.

**(iii)** Is the matrix $A$ diagonalizable? Explain.

**(iv)** Find all eigenvalues of the matrix $A^2$.

**Bonus Problem 4 (20 pts.)** Find a linear polynomial which is the best least squares fit to the following data:

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | $-3$ | $-2$ | 1 | 2 | 5 |

**Bonus Problem 5 (20 pts.)** Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space $V$ to a vector space $W$. Show that

$$\dim \mathrm{Range}(L) + \dim \ker(L) = \dim V.$$

**Problem 1 (20 pts.)** Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the vector space of $2 \times 2$ matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Find the matrix of the operator $L$ with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $M_L$ denote the desired matrix.

By definition, $M_L$ is a $4 \times 4$ matrix whose columns are coordinates of the matrices $L(E_1), L(E_2), L(E_3), L(E_4)$ with respect to the basis $E_1, E_2, E_3, E_4$.

$$L(E_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1E_1 + 2E_2 + 0E_3 + 0E_4,$$

$$L(E_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} = 3E_1 + 4E_2 + 0E_3 + 0E_4,$$

$$L(E_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = 0E_1 + 0E_2 + 1E_3 + 2E_4,$$

$$L(E_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = 0E_1 + 0E_2 + 3E_3 + 4E_4.$$

It follows that

$$M_L = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{pmatrix}.$$

Thus the relation

$$\begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

is equivalent to the relation

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

**Problem 2 (30 pts.)** Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

**(i)** Find an orthonormal basis for $V$.

First we apply the Gram-Schmidt orthogonalization process to vectors $\mathbf{x}_1, \mathbf{x}_2$ and obtain an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2$ for the subspace $V$:

$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1, 1)$,

$\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = (1, 0, 3, 0) - \dfrac{4}{4}(1, 1, 1, 1) = (0, -1, 2, -1)$.

Then we normalize vectors $\mathbf{v}_1, \mathbf{v}_2$ to obtain an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2$ for $V$:

$\|\mathbf{v}_1\| = 2 \implies \mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}(1, 1, 1, 1)$

$\|\mathbf{v}_2\| = \sqrt{6} \implies \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}(0, -1, 2, -1)$

**Problem 2 (30 pts.)** Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

**(ii)** Find an orthonormal basis for the orthogonal complement $V^\perp$.

Since the subspace $V$ is spanned by vectors $(1, 1, 1, 1)$ and $(1, 0, 3, 0)$, it is the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

Then the orthogonal complement $V^\perp$ is the nullspace of $A$. To find the nullspace, we convert the matrix $A$ to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

Hence a vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ belongs to $V^\perp$ if and only if

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{cases} x_1 + 3x_3 = 0 \\ x_2 - 2x_3 + x_4 = 0 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} x_1 = -3x_3 \\ x_2 = 2x_3 - x_4 \end{cases}$$

The general solution of the system is $(x_1, x_2, x_3, x_4) =$
$= (-3t, 2t - s, t, s) = t(-3, 2, 1, 0) + s(0, -1, 0, 1)$, where $t, s \in \mathbb{R}$.

It follows that $V^\perp$ is spanned by vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$.

The vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$ form a basis for the subspace $V^\perp$.

It remains to orthogonalize and normalize this basis:

$$\mathbf{v}_3 = \mathbf{x}_3 = (0, -1, 0, 1),$$

$$\mathbf{v}_4 = \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3} \mathbf{v}_3 = (-3, 2, 1, 0) - \frac{-2}{2}(0, -1, 0, 1)$$

$$= (-3, 1, 1, 1),$$

$$\|\mathbf{v}_3\| = \sqrt{2} \implies \mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{2}}(0, -1, 0, 1),$$

$$\|\mathbf{v}_4\| = \sqrt{12} = 2\sqrt{3} \implies \mathbf{w}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1).$$

Thus the vectors $\mathbf{w}_3 = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$ and $\mathbf{w}_4 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$ form an orthonormal basis for $V^\perp$.

**Problem 2 (30 pts.)** Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

**(i)** Find an orthonormal basis for $V$.

**(ii)** Find an orthonormal basis for the orthogonal complement $V^\perp$.

*Alternative solution:* First we extend the set $\mathbf{x}_1, \mathbf{x}_2$ to a basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ for $\mathbb{R}^4$. Then we orthogonalize and normalize the latter. This yields an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ for $\mathbb{R}^4$.

By construction, $\mathbf{w}_1, \mathbf{w}_2$ is an orthonormal basis for $V$. It follows that $\mathbf{w}_3, \mathbf{w}_4$ is an orthonormal basis for $V^\perp$.

The set $\mathbf{x}_1 = (1, 1, 1, 1)$, $\mathbf{x}_2 = (1, 0, 3, 0)$ can be extended to a basis for $\mathbb{R}^4$ by adding two vectors from the standard basis.

For example, we can add vectors $\mathbf{e}_3 = (0, 0, 1, 0)$ and $\mathbf{e}_4 = (0, 0, 0, 1)$. To show that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{e}_3, \mathbf{e}_4$ is indeed a basis for $\mathbb{R}^4$, we check that the matrix whose rows are these vectors is nonsingular:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0.$$

To orthogonalize the basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{e}_3, \mathbf{e}_4$, we apply the Gram-Schmidt process:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1, 1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1, 0, 3, 0) - \tfrac{4}{4}(1, 1, 1, 1) = (0, -1, 2, -1),$$

$$\mathbf{v}_3 = \mathbf{e}_3 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (0, 0, 1, 0) - \tfrac{1}{4}(1, 1, 1, 1) -$$
$$-\tfrac{2}{6}(0, -1, 2, -1) = \left(-\tfrac{1}{4}, \tfrac{1}{12}, \tfrac{1}{12}, \tfrac{1}{12}\right) = \tfrac{1}{12}(-3, 1, 1, 1),$$

$$\mathbf{v}_4 = \mathbf{e}_4 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 = (0, 0, 0, 1) -$$
$$-\tfrac{1}{4}(1, 1, 1, 1) - \tfrac{-1}{6}(0, -1, 2, -1) - \tfrac{1/12}{1/12} \cdot \tfrac{1}{12}(-3, 1, 1, 1) =$$
$$= \left(0, -\tfrac{1}{2}, 0, \tfrac{1}{2}\right) = \tfrac{1}{2}(0, -1, 0, 1).$$

It remains to normalize vectors $\mathbf{v}_1 = (1, 1, 1, 1)$,
$\mathbf{v}_2 = (0, -1, 2, -1)$, $\mathbf{v}_3 = \frac{1}{12}(-3, 1, 1, 1)$, $\mathbf{v}_4 = \frac{1}{2}(0, -1, 0, 1)$:

$$\|\mathbf{v}_1\| = 2 \implies \mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}(1, 1, 1, 1)$$

$$\|\mathbf{v}_2\| = \sqrt{6} \implies \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}(0, -1, 2, -1)$$

$$\|\mathbf{v}_3\| = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} \implies \mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$$

$$\|\mathbf{v}_4\| = \frac{1}{\sqrt{2}} \implies \mathbf{w}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$$

Thus $\mathbf{w}_1, \mathbf{w}_2$ is an orthonormal basis for $V$ while $\mathbf{w}_3, \mathbf{w}_4$ is an orthonormal basis for $V^\perp$.

**Problem 3 (30 pts.)** Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

**(i)** Find all eigenvalues of the matrix $A$.

The eigenvalues of $A$ are roots of the characteristic equation $\det(A - \lambda I) = 0$. We obtain that

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 & 0 \\ 1 & 1 - \lambda & 1 \\ 0 & 2 & 1 - \lambda \end{vmatrix}$$

$$= (1 - \lambda)^3 - 2(1 - \lambda) - 2(1 - \lambda) = (1 - \lambda)\big((1 - \lambda)^2 - 4\big)$$

$$= (1 - \lambda)\big((1 - \lambda) - 2\big)\big((1 - \lambda) + 2\big) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix $A$ has three eigenvalues: $-1$, $1$, and $3$.

**Problem 3 (30 pts.)** Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

**(ii)** For each eigenvalue of $A$, find an associated eigenvector.

An eigenvector $\mathbf{v} = (x, y, z)$ of the matrix $A$ associated with an eigenvalue $\lambda$ is a nonzero solution of the vector equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 - \lambda & 2 & 0 \\ 1 & 1 - \lambda & 1 \\ 0 & 2 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve the equation, we convert the matrix $A - \lambda I$ to reduced row echelon form.

First consider the case $\lambda = -1$. The row reduction yields

$$A + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A + I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)$ is an eigenvector of $A$ associated with the eigenvalue $-1$.

Secondly, consider the case $\lambda = 1$. The row reduction yields

$$A - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, 0, 1)$ is an eigenvector of $A$ associated with the eigenvalue 1.

Finally, consider the case $\lambda = 3$. The row reduction yields

$$A - 3I = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - 3I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = t$, $y = t$, $z = t$, where $t \in \mathbb{R}$.
In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of $A$ associated with the eigenvalue 3.

**Problem 3 (30 pts.)** Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

**(iii)** Is the matrix $A$ diagonalizable? Explain.

The matrix $A$ is diagonalizable, i.e., there exists a basis for $\mathbb{R}^3$ formed by its eigenvectors.

Namely, the vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix $A$ belonging to distinct eigenvalues. Therefore these vectors are linearly independent. It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

Alternatively, the existence of a basis for $\mathbb{R}^3$ consisting of eigenvectors of $A$ already follows from the fact that the matrix $A$ has three distinct eigenvalues.

**Problem 3 (30 pts.)** Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

**(iv)** Find all eigenvalues of the matrix $A^2$.

We have that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The columns of $U$ are eigenvectors of $A$ belonging to the eigenvalues $-1$, $1$, and $3$, respectively.

Then $A^2 = UBU^{-1}UBU^{-1} = UB^2U^{-1}$, that is, the matrix $A^2$ is similar to the diagonal matrix $B^2 = \operatorname{diag}(1, 1, 9)$.

Similar matrices have the same characteristic polynomial, hence they have the same eigenvalues. Thus the eigenvalues of $A^2$ are the same as the eigenvalues of $B^2$: 1 and 9.

**Bonus Problem 4 (20 pts.)** Find a linear polynomial which is the best least squares fit to the following data:

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $-3$ | $-2$ | $1$ | $2$ | $5$ |

We are looking for a function $f(x) = c_1 + c_2 x$, where $c_1, c_2$ are unknown coefficients. The data of the problem give rise to an overdetermined system of linear equations in variables $c_1$ and $c_2$:

$$\begin{cases} c_1 - 2c_2 = -3, \\ c_1 - c_2 = -2, \\ c_1 = 1, \\ c_1 + c_2 = 2, \\ c_1 + 2c_2 = 5. \end{cases}$$

This system is inconsistent.

We can represent the system as a matrix equation $A\mathbf{c} = \mathbf{y}$, where

$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

The least squares solution $\mathbf{c}$ of the above system is a solution of the normal system $A^T A \mathbf{c} = A^T \mathbf{y}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\iff \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 20 \end{pmatrix} \iff \begin{cases} c_1 = 3/5 \\ c_2 = 2 \end{cases}$$

Thus the function $f(x) = \frac{3}{5} + 2x$ is the best least squares fit to the above data among linear polynomials.
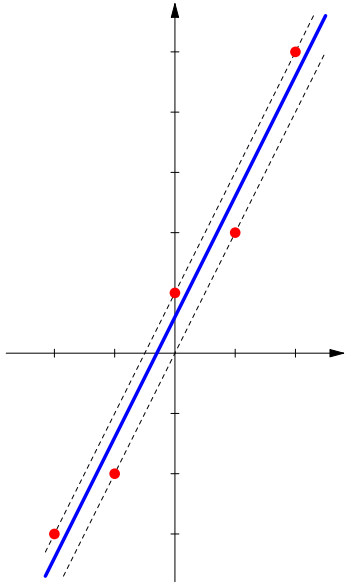
**Bonus Problem 5 (20 pts.)** Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space $V$ to a vector space $W$. Show that $\dim \mathrm{Range}(L) + \dim \ker(L) = \dim V$.

The kernel $\ker(L)$ is a subspace of $V$. It is finite-dimensional since the vector space $V$ is.

Take a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ for the subspace $\ker(L)$, then extend it to a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ for the entire space $V$.

**Claim** Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \ldots, L(\mathbf{u}_m)$ form a basis for the range of $L$.

Assuming the claim is proved, we obtain

$$\dim \mathrm{Range}(L) = m, \quad \dim \ker(L) = k, \quad \dim V = k + m.$$

**Claim** Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \ldots, L(\mathbf{u}_m)$ form a basis for the range of $L$.

*Proof (spanning):* Any vector $\mathbf{w} \in \mathrm{Range}(L)$ is represented as $\mathbf{w} = L(\mathbf{v})$, where $\mathbf{v} \in V$. Then

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k + \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \cdots + \beta_m \mathbf{u}_m$$

for some $\alpha_i, \beta_j \in \mathbb{R}$. It follows that

$$\mathbf{w} = L(\mathbf{v}) = \alpha_1 L(\mathbf{v}_1) + \cdots + \alpha_k L(\mathbf{v}_k) + \beta_1 L(\mathbf{u}_1) + \cdots + \beta_m L(\mathbf{u}_m)$$

$$= \beta_1 L(\mathbf{u}_1) + \cdots + \beta_m L(\mathbf{u}_m).$$

Note that $L(\mathbf{v}_i) = \mathbf{0}$ since $\mathbf{v}_i \in \ker(L)$.

Thus $\mathrm{Range}(L)$ is spanned by the vectors $L(\mathbf{u}_1), \ldots, L(\mathbf{u}_m)$.

**Claim** Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \ldots, L(\mathbf{u}_m)$ form a basis for the range of $L$.

*Proof (linear independence):* Suppose that
$$t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \cdots + t_m L(\mathbf{u}_m) = \mathbf{0}$$
for some $t_i \in \mathbb{R}$. Let $\mathbf{u} = t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_m\mathbf{u}_m$. Since

$$L(\mathbf{u}) = t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \cdots + t_m L(\mathbf{u}_m) = \mathbf{0},$$

the vector $\mathbf{u}$ belongs to the kernel of $L$. Therefore $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k$ for some $s_j \in \mathbb{R}$. It follows that

$$t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_m\mathbf{u}_m - s_1\mathbf{v}_1 - s_2\mathbf{v}_2 - \cdots - s_k\mathbf{v}_k = \mathbf{u} - \mathbf{u} = \mathbf{0}.$$

Linear independence of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_1, \ldots, \mathbf{u}_m$ implies that $t_1 = \cdots = t_m = 0$ (as well as $s_1 = \cdots = s_k = 0$). Thus the vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \ldots, L(\mathbf{u}_m)$ are linearly independent.