Math 304-504

Linear algebra

Lecture 35: Symmetric and orthogonal matrices.

Problem. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Find e^{tA}.

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots$$

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix},$$

where $a(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots = \cos t$,

$b(t) = -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \cdots = -\sin t$,

$c(t) = -b(t) = \sin t, \quad d(t) = a(t) = \cos t.$

Thus $e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$.

Let $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

- A is the matrix of rotation by angle ϕ
- $A_\phi A_\psi = A_{\phi+\psi}$
- $A_\phi^T = A_{-\phi}$
- $A_\phi^{-1} = A_{-\phi} = A_\phi^T$
- Columns of A_ϕ form an orthonormal basis for $\mathbb{R}^2$
- Rows of A_ϕ form an orthonormal basis for $\mathbb{R}^2$

Proposition For any $n \times n$ matrix A and any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\boxed{A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^T\mathbf{y}.}$

Proof: $A\mathbf{x} \cdot \mathbf{y} = \mathbf{y}^T A\mathbf{x} = (\mathbf{y}^T A\mathbf{x})^T = \mathbf{x}^T A^T \mathbf{y} =$
$= A^T\mathbf{y} \cdot \mathbf{x} = \mathbf{x} \cdot A^T\mathbf{y}.$

Definition. An $n \times n$ matrix A is called

- **symmetric** if $A^T = A$;
- **orthogonal** if $AA^T = A^T A = I$, that is, if $A^T = A^{-1}$;
- **normal** if $AA^T = A^T A$.

Clearly, symmetric and orthogonal matrices are normal.

Theorem If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of a symmetric matrix A associated with different eigenvalues, then $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof: Suppose $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$, where $\lambda \neq \mu$. Then $A\mathbf{x} \cdot \mathbf{y} = \lambda(\mathbf{x} \cdot \mathbf{y})$, $\mathbf{x} \cdot A\mathbf{y} = \mu(\mathbf{x} \cdot \mathbf{y})$.
But $A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^T\mathbf{y} = \mathbf{x} \cdot A\mathbf{y}$.
Thus $\lambda(\mathbf{x} \cdot \mathbf{y}) = \mu(\mathbf{x} \cdot \mathbf{y}) \implies \mathbf{x} \cdot \mathbf{y} = 0$.

Theorem Suppose A is a symmetric $n \times n$ matrix. Then **(a)** all eigenvalues of A are real;
(b) there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Example. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

- A is symmetric.
- A has three eigenvalues: 0, 2, and 3.
- Associated eigenvectors are $\mathbf{v}_1 = (-1, 0, 1)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (0, 1, 0)$.
- Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal basis for $\mathbb{R}^3$.

Theorem If A is a normal matrix then $N(A) = N(A^T)$ (that is, $A\mathbf{x} = \mathbf{0} \iff A^T\mathbf{x} = \mathbf{0}$).

Proof: $A\mathbf{x} = \mathbf{0} \iff A\mathbf{x} \cdot A\mathbf{x} = 0 \iff \mathbf{x} \cdot A^T A\mathbf{x} = 0$
$\iff \mathbf{x} \cdot AA^T\mathbf{x} = 0 \iff A^T\mathbf{x} \cdot A^T\mathbf{x} = 0 \iff A^T\mathbf{x} = \mathbf{0}$.

Proposition If a matrix A is normal, so are matrices $A - \lambda I$, $\lambda \in \mathbb{R}$.

Proof: Let $B = A - \lambda I$, where $\lambda \in \mathbb{R}$. Then
$B^T = (A - \lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I$.
We have $BB^T = (A - \lambda I)(A^T - \lambda I) = AA^T - \lambda A - \lambda A^T + \lambda^2 I$,
$B^T B = (A^T - \lambda I)(A - \lambda I) = A^T A - \lambda A - \lambda A^T + \lambda^2 I$.
Hence $AA^T = A^T A \implies BB^T = B^T B$.

Thus any normal matrix A shares with A^T all real eigenvalues and the corresponding eigenvectors.
How about complex eigenvalues?

Dot product of complex vectors

Dot product of real vectors

$\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Dot product of complex vectors

$\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{C}^n$:

$$\mathbf{x} \cdot \mathbf{y} = x_1 \overline{y_1} + x_2 \overline{y_2} + \cdots + x_n \overline{y_n}.$$

If $z = r + it$ $(r, t \in \mathbb{R})$ then $\overline{z} = r - it$,

$z\overline{z} = r^2 + t^2 = |z|^2$.

Hence $\mathbf{x} \cdot \mathbf{x} = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq 0$.

Also, $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Since $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$, it follows that

$\mathbf{y} \cdot \mathbf{x} = \overline{\mathbf{x} \cdot \mathbf{y}}$.

Definition. Let V be a complex vector space. A function $\beta : V \times V \to \mathbb{C}$, denoted $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$, is called an **inner product** on V if

(i) $\langle \mathbf{x}, \mathbf{y} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)

(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ (conjugate symmetry)

(iii) $\langle r\mathbf{x}, \mathbf{y} \rangle = r\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity)

(iv) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (additivity)

$\langle \mathbf{x}, \mathbf{y} \rangle$ is complex-linear as a function of $\mathbf{x}$.
The dependence on the second argument is called *half-linearity*: $\langle \mathbf{x}, \lambda\mathbf{y} + \mu\mathbf{z} \rangle = \overline{\lambda}\langle \mathbf{x}, \mathbf{y} \rangle + \overline{\mu}\langle \mathbf{x}, \mathbf{z} \rangle$.

Example. $\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx$,
$f, g \in C([a, b], \mathbb{C})$.

Theorem Suppose A is a normal matrix. Then for any $\mathbf{x} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ one has

$$A\mathbf{x} = \lambda\mathbf{x} \iff A^T\mathbf{x} = \overline{\lambda}\mathbf{x}.$$

Corollary All eigenvalues of a symmetric matrix are real. All eigenvalues λ of an orthogonal matrix satisfy $|\lambda| = 1$.

Theorem Suppose A is a normal $n \times n$ matrix. Then there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A.

Orthogonal matrices

Theorem Given an $n \times n$ matrix A, the following conditions are equivalent:
(i) A is orthogonal: $A^T = A^{-1}$;
(ii) columns of A form an orthonormal basis for $\mathbb{R}^n$;
(iii) rows of A form an orthonormal basis for $\mathbb{R}^n$.

[Entries of the matrix $A^T A$ are the dot products of columns of A. Entries of AA^T are the dot products of rows of A.]

Thus an orthogonal matrix is the transition matrix from one orthonormal basis to another.