Math 304–504

Linear Algebra

Lecture 3:
Gauss-Jordan reduction.
Applications of systems of linear equations.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix ($m \times n$) and column vector of the right-hand sides ($m \times 1$):

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Augmented $m \times (n+1)$ *matrix:*

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

Solution of a system of linear equations splits into two parts: **(A)** elimination and **(B)** back substitution.

Both parts can be done by applying a finite number of **elementary operations**.

Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the **augmented matrix**.

In terms of the augmented matrix, the elementary operations are **elementary row operations**.

Elementary row operations:

(1) to multiply a row by some $r \neq 0$;
(2) to add a row multiplied by some $r \in \mathbb{R}$ to another row;
(3) to interchange two rows.

Remark. The rows are added and multiplied by scalars as vectors (namely, **row vectors**):

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix},$$

where $\mathbf{v}_i = (a_{i1}\ a_{i2}\ \dots\ a_{in} \,|\, b_i)$ is a row vector.

Operation 1: to multiply the ith row by $r \neq 0$:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Operation 2: to add the ith row multiplied by r to the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \to \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j + r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Operation 3: to interchange the ith row with the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

The goal of the Gaussian elimination is to convert the augmented matrix into **row echelon form**:

$$\left[\begin{array}{cccccccccc|c}
\boxed{} & * & * & * & * & * & * & * & * & * & * \\
 & \boxed{} & * & * & * & * & * & * & * & * & * \\
 & & & & \boxed{} & * & * & * & * & * & * \\
 & & & & & & \boxed{} & * & * & * & * \\
 & & & & & & & \boxed{} & * & * & * \\
 & & & & & & & & & & \\
 & & & & & & & & & &
\end{array}\right]$$

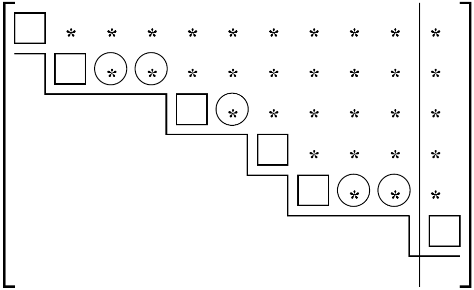
- all the entries below the staircase line are zero;
- boxed entries, called **pivot** or **lead entries**, are equal to 1;
- each circled star correspond to a free variable.

Strict triangular form is a particular case of row echelon form that can occur for systems of n equations in n variables:

$$\begin{bmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{bmatrix}$$

Matrix of coefficients

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.



Inconsistent system

The goal of the **Gauss-Jordan reduction** is to convert the augmented matrix into **reduced row echelon form**:

$$\left[\begin{array}{cccccccccc|c}
\boxed{1} & 0 & * & * & 0 & * & 0 & 0 & * & * & * \\
 & \boxed{1} & * & * & 0 & * & 0 & 0 & * & * & * \\
 & & & & \boxed{1} & * & 0 & 0 & * & * & * \\
 & & & & & & \boxed{1} & 0 & * & * & * \\
 & & & & & & & \boxed{1} & * & * & * \\
 & & & & & & & & & & \\
 & & & & & & & & & &
\end{array}\right]$$

- all entries below the staircase line are zero;
- each pivot entry is 1, the other entries in its column are zero;
- each circled star correspond to a free variable.

Example 1. $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases}$

Augmented matrix: $\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 2 & 3 & 6 \end{array}\right)$

The matrix is in row echelon form. To convert it into reduced row echelon form, add -2 times the 2nd row to the 1st row:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & -2 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array}\right)$$

x_3 and x_4 are free variables

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

System of linear equations:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases}$$

General solution:

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} \qquad (t, s \in \mathbb{R})$$

Example 2. $\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases} \quad (a \in \mathbb{R})$

The system is **homogeneous** (all right-hand sides are zeros). Therefore it is consistent ($x = y = z = 0$ is a solution).

Augmented matrix: $\left(\begin{array}{ccr|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array}\right)$

Since the 1st row cannot serve as a pivotal one, we interchange it with the 2nd row:

$$\left(\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array}\right)$$

Now we can start the elimination. First add -1 times the 1st row to the 3rd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array}\right)$$

The 2nd row is our new pivotal row. Add -1 times the 2nd row to the 3rd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right)$$

At this point row reduction is divided into two cases.

Case 1: $a \neq 1$. In this case, multiply the 3rd row by $(a-1)^{-1}$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

The matrix is converted into row echelon form.
We proceed towards reduced row echelon form.

Add -3 times the 3rd row to the 2nd row:

$$\left(\begin{array}{ccr|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccr|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Add 2 times the 3rd row to the 1st row:

$$\left(\begin{array}{ccr|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Finally, add -1 times the 2nd row to the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

Thus $x = y = z = 0$ is the only solution.

Case 2: $a = 1$. In this case, the matrix is already in row echelon form:

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

To get reduced row echelon form, add -1 times the 2nd row to the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & -5 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z is a free variable.

$$\begin{cases} x - 5z = 0 \\ y + 3z = 0 \end{cases} \iff \begin{cases} x = 5z \\ y = -3z \end{cases}$$

System of linear equations:

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases}$$

Solution: If $a \neq 1$ then $(x, y, z) = (0, 0, 0)$;
if $a = 1$ then $(x, y, z) = (5t, -3t, t)$, $t \in \mathbb{R}$.

Applications

Problem 1 Find the point of intersection of the lines $x - y = -2$ and $2x + 3y = 6$ in $\mathbb{R}^2$.

$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases}$$

Problem 2 Find the point of intersection of the planes $x - y = 2$, $2x - y - z = 3$, and $x + y + z = 6$ in $\mathbb{R}^3$.

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

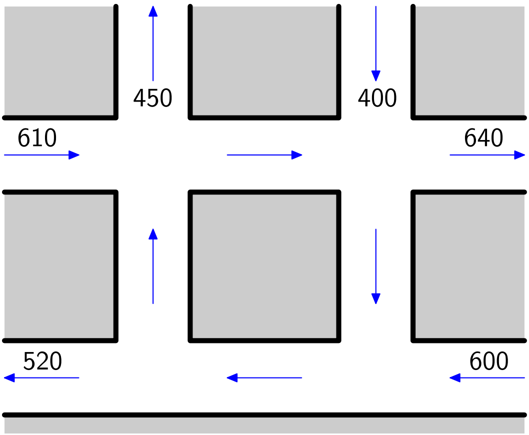
Method of undetermined coefficients often involves solving systems of linear equations.

Problem 3. Find a quadratic polynomial $p(x)$ such that $p(1) = 4$, $p(2) = 3$, and $p(3) = 4$.

Suppose that $p(x) = ax^2 + bx + c$. Then $p(1) = a + b + c$, $p(2) = 4a + 2b + c$, $p(3) = 9a + 3b + c$.

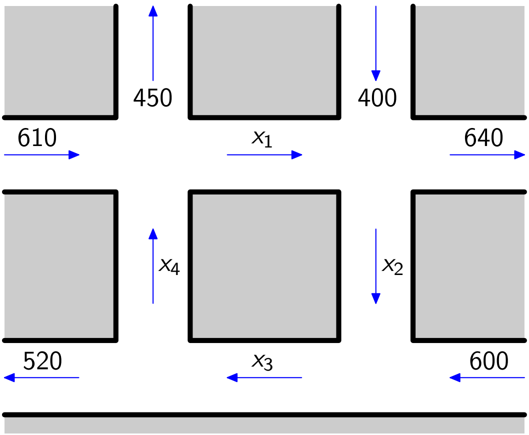
$$\begin{cases} a + b + c = 4 \\ 4a + 2b + c = 3 \\ 9a + 3b + c = 4 \end{cases}$$

Traffic flow



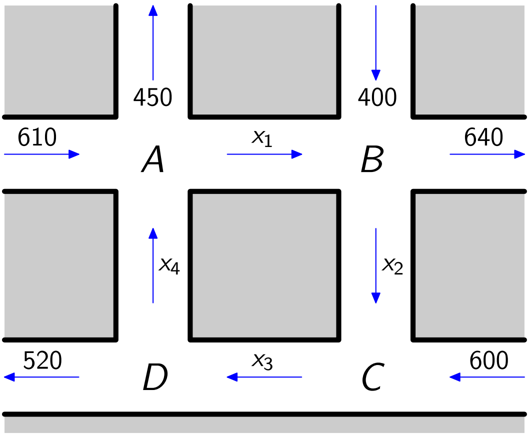
Problem. Determine the amount of traffic between each of the four intersections.

Traffic flow



$$x_1 = ?, \quad x_2 = ?, \quad x_3 = ?, \quad x_4 = ?$$

Traffic flow



At each intersection, the incoming traffic has to match the outgoing traffic.

Intersection A: $\quad x_4 + 610 = x_1 + 450$

Intersection B: $\quad x_1 + 400 = x_2 + 640$

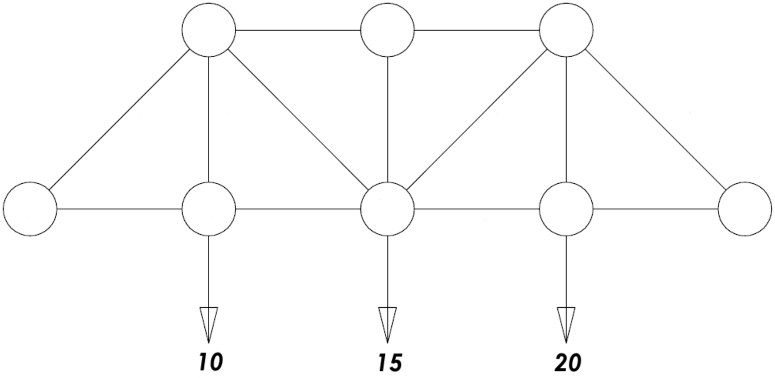
Intersection C: $\quad x_2 + 600 = x_3$

Intersection D: $\quad x_3 = x_4 + 520$

$$\begin{cases} x_4 + 610 = x_1 + 450 \\ x_1 + 400 = x_2 + 640 \\ x_2 + 600 = x_3 \\ x_3 = x_4 + 520 \end{cases}$$

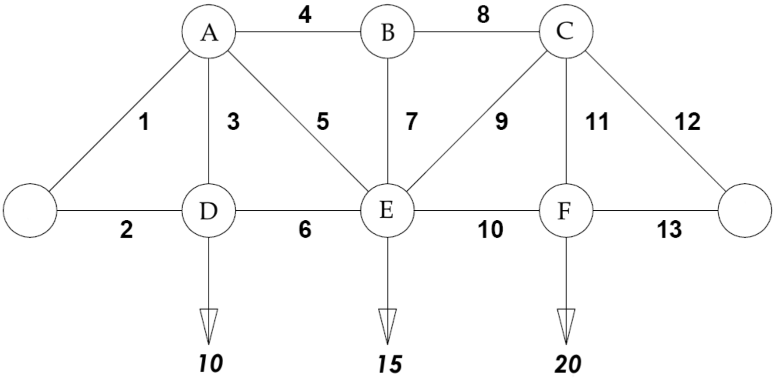
$$\iff \begin{cases} -x_1 + x_4 = -160 \\ x_1 - x_2 = 240 \\ x_2 - x_3 = -600 \\ x_3 - x_4 = 520 \end{cases}$$

Stress analysis of a truss



Problem. Assume that the leftmost and rightmost joints are fixed. Find the forces acting on each member of the truss.

Truss bridge



Let $|f_k|$ be the magnitude of the force in the kth member. $f_k > 0$ if the member is under tension. $f_k < 0$ if the member is under compression.

Static equilibrium at the joint A:

horizontal projection: $-\frac{1}{\sqrt{2}}f_1 + f_4 + \frac{1}{\sqrt{2}}f_5 = 0$

vertical projection: $-\frac{1}{\sqrt{2}}f_1 - f_3 - \frac{1}{\sqrt{2}}f_5 = 0$

Static equilibrium at the joint B:

horizontal projection: $-f_4 + f_8 = 0$

vertical projection: $-f_7 = 0$

Static equilibrium at the joint C:

horizontal projection: $-f_8 - \frac{1}{\sqrt{2}}f_9 + \frac{1}{\sqrt{2}}f_{12} = 0$

vertical projection: $-\frac{1}{\sqrt{2}}f_9 - f_{11} - \frac{1}{\sqrt{2}}f_{12} = 0$

Static equilibrium at the joint D:

horizontal projection: $-f_2 + f_6 = 0$

vertical projection: $f_3 - 10 = 0$

Static equilibrium at the joint E:

horizontal projection: $-\frac{1}{\sqrt{2}} f_5 - f_6 + \frac{1}{\sqrt{2}} f_9 + f_{10} = 0$

vertical projection: $\frac{1}{\sqrt{2}} f_5 + f_7 + \frac{1}{\sqrt{2}} f_9 - 15 = 0$

Static equilibrium at the joint F:

horizontal projection: $-f_{10} + f_{13} = 0$

vertical projection: $f_{11} - 20 = 0$

$$
\begin{cases}
-\frac{1}{\sqrt{2}}f_1 + f_4 + \frac{1}{\sqrt{2}}f_5 = 0 \\
-\frac{1}{\sqrt{2}}f_1 - f_3 - \frac{1}{\sqrt{2}}f_5 = 0 \\
-f_4 + f_8 = 0 \\
-f_7 = 0 \\
-f_8 - \frac{1}{\sqrt{2}}f_9 + \frac{1}{\sqrt{2}}f_{12} = 0 \\
-\frac{1}{\sqrt{2}}f_9 - f_{11} - \frac{1}{\sqrt{2}}f_{12} = 0 \\
-f_2 + f_6 = 0 \\
f_3 = 10 \\
-\frac{1}{\sqrt{2}}f_5 - f_6 + \frac{1}{\sqrt{2}}f_9 + f_{10} = 0 \\
\frac{1}{\sqrt{2}}f_5 + f_7 + \frac{1}{\sqrt{2}}f_9 = 15 \\
-f_{10} + f_{13} = 0 \\
f_{11} = 20
\end{cases}
$$