Math 304–504

Linear Algebra

Lecture 7:
Elementary matrices.
Determinants.

Inverse matrix

Definition. Let A be an $n \times n$ matrix. The **inverse** of A is an $n \times n$ matrix, denoted A^{-1}, such that

$$\boxed{AA^{-1} = A^{-1}A = I.}$$

If A^{-1} exists then the matrix A is called **invertible**. Otherwise A is called **singular**.

Inverting diagonal matrices

Theorem A diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

$$\begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n \end{pmatrix}^{-1} = \begin{pmatrix} d_1^{-1} & 0 & \ldots & 0 \\ 0 & d_2^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n^{-1} \end{pmatrix}$$

Inverting 2×2 matrices

Definition. The **determinant** of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$.

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$.

If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Fundamental results on inverse matrices

Theorem 1 Given a square matrix A, the following are equivalent:

(i) A is invertible;
(ii) $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$;
(iii) the row echelon form of A has no zero rows;
(iv) the reduced row echelon form of A is the identity matrix.

Theorem 2 Suppose that a sequence of elementary row operations converts a matrix A into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix A^{-1}.

Theorem 3 For any $n\times n$ matrices A and B,

$$BA = I \iff AB = I.$$

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$.

A convenient way to compute the inverse matrix A^{-1} is to merge the matrices A and I into one 3×6 matrix $(A\,|\,I)$, and apply elementary row operations to this new matrix.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A\,|\,I) = \left(\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(A\,|\,I) = \left(\begin{array}{rrr|rrr} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

As soon as the left half of the 3×6 matrix is converted to the identity matrix, we have got the inverse matrix A^{-1} in the right half.

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & 0 & 1 & -\frac{3}{5} & 1 & -\frac{2}{5} \end{array}\right)$$

Thus $$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Why does it work?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1+3a_1 & b_2+3a_2 & b_3+3a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Proposition Any elementary row operation can be simulated as left multiplication by a certain matrix.

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & O & \\ & & 1 & & & & \\ & & & r & & & \\ & & & & 1 & & \\ & O & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \quad \text{row } \#i$$

To obtain the matrix EA from A, multiply the ith row by r. To obtain the matrix AE from A, multiply the ith column by r.

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & O & \\ 0 & \cdots & 1 & & & & \\ \vdots & & \vdots & \ddots & & & \\ 0 & \cdots & r & \cdots & 1 & & \\ \vdots & & \vdots & & \vdots & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \begin{matrix} \\ \\ \text{row } \#i \\ \\ \text{row } \#j \\ \\ \\ \end{matrix}$$

To obtain the matrix EA from A, add r times the ith row to the jth row. To obtain the matrix AE from A, add r times the jth column to the ith column.

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & O & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & O & & & & & 1 \end{pmatrix} \quad \begin{matrix} \\ \\ \text{row } \#i \\ \\ \text{row } \#j \\ \\ \\ \end{matrix}$$

To obtain the matrix EA from A, interchange the ith row with the jth row. To obtain AE from A, interchange the ith column with the jth column.

Why does it work?

Assume that a square matrix A can be converted to the identity matrix by a sequence of elementary row operations. Then

$$E_k E_{k-1} \dots E_2 E_1 A = I,$$

where $E_1, E_2, \dots, E_k$ are elementary matrices corresponding to those operations.

Applying the same sequence of operations to the identity matrix, we obtain the matrix

$$B = E_k E_{k-1} \dots E_2 E_1 I = E_k E_{k-1} \dots E_2 E_1.$$

Thus $BA = I$, which implies that $B = A^{-1}$.

Problem Solve the matrix equation $XA + B = X$,
where $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix}$.

Since B is a 2×2 matrix, it follows that XA and X are also 2×2 matrices.

$$XA + B = X \iff X - XA = B$$
$$\iff X(I - A) = B \iff X = B(I - A)^{-1}$$

provided that $I - A$ is an invertible matrix.

$I - A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$,

$$I - A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix},$$

$$\det(I - A) = (-3) \cdot 0 - 2 \cdot (-1) = 2,$$

$$(I - A)^{-1} = \frac{1}{2} \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix},$$

$$X = B(I - A)^{-1} = \begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & -16 \\ 0 & -6 \end{pmatrix} = \begin{pmatrix} 1 & -8 \\ 0 & -3 \end{pmatrix}.$$

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ij})_{1 \le i,j \le n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A = 0$ if and only if the matrix A is singular.

Definition in low dimensions

Definition. $\det(a) = a$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+: \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$-: \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Examples: 2×2 matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = -12,$$

$$\begin{vmatrix} -2 & 5 \\ 0 & 3 \end{vmatrix} = -6, \qquad \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14,$$

$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1, \qquad \begin{vmatrix} 0 & 0 \\ 4 & 1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 2 & 1 \\ 8 & 4 \end{vmatrix} = 0.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \left(\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right)$$

$$+ \left(\begin{array}{ccc|cc} \boxed{1} & \boxed{2} & \boxed{3} & * & * \\ * & \boxed{1} & \boxed{2} & \boxed{3} & * \\ * & * & \boxed{1} & \boxed{2} & \boxed{3} \end{array}\right) \quad - \left(\begin{array}{ccc|cc} * & * & \boxed{1} & \boxed{2} & \boxed{3} \\ * & \boxed{1} & \boxed{2} & \boxed{3} & * \\ \boxed{1} & \boxed{2} & \boxed{3} & * & * \end{array}\right)$$

This rule works **only** for 3×3 matrices!

Examples: 3×3 matrices

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3 -$$

$$- 0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = 4 - 9 = -5,$$

$$\begin{vmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0 -$$

$$- 6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0 = 1 \cdot 2 \cdot 3 = 6.$$

General definition

The general definition of the determinant is quite complicated as there are no simple explicit formula.

There are several approaches to defining determinants.

Approach 1 (original): an explicit (but very complicated) formula.

Approach 2 (axiomatic): we formulate properties that the determinant should have.

Approach 3 (inductive): the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n-1)\times(n-1)$ matrices.

$\mathcal{M}_n(\mathbb{R})$: the set of $n \times n$ matrices with real entries.

Theorem There exists a unique function $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ (called the determinant) with the following properties:

- if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;
- if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;
- if we interchange two rows of a matrix, the determinant changes its sign;
- $\det I = 1$.

Corollary $\det A = 0$ if and only if the matrix A is singular.

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$, $\det A = ?$

We have transformed the matrix A into the identity matrix using elementary row operations.

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by $-1/2$,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by $-2/5$,
- add $-3/2$ times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$, $\det A = ?$

We have transformed the matrix A into the identity matrix using elementary row operations.

These included two row multiplications, by $-1/2$ and by $-2/5$, and one row exchange.

It follows that

$$\det I = -\left(-\frac{1}{2}\right)\left(-\frac{2}{5}\right)\det A = -\frac{1}{5}\det A.$$

Hence $\det A = -5 \det I = -5.$