Math 304–504

Linear Algebra

Lecture 8:
Properties of determinants.

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ij})_{1 \le i,j \le n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A = 0$ if and only if the matrix A is singular.

Definition in low dimensions

Definition. $\det(a) = a$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+: \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$-: \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Example

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3 -$$

$$- 0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = -5$$

General definition

There are several approaches to defining determinants.

Approach 1 (original): an explicit (but very complicated) formula.

Approach 2 (axiomatic): we formulate properties that the determinant should have.

Approach 3 (inductive): the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n-1)\times(n-1)$ matrices.

$\mathcal{M}_n(\mathbb{R})$: the set of $n \times n$ matrices with real entries.

Theorem There exists a unique function $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ (called the determinant) with the following properties:

- if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;
- if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;
- if we interchange two rows of a matrix, the determinant changes its sign;
- $\det I = 1$.

Corollary $\det A = 0$ if and only if the matrix A is singular.

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$, $\det A = ?$

We had transformed the matrix A into the identity matrix using elementary row operations.

These included two row multiplications, by $-1/2$ and by $-2/5$, and one row exchange.

It follows that

$$\det I = -\left(-\tfrac{1}{2}\right)\left(-\tfrac{2}{5}\right)\det A = -\tfrac{1}{5}\det A.$$

Hence $\det A = -5\det I = -5.$

Permutations

Definition. A *permutation* on the set $\{1, 2, \ldots, n\}$ is a one-to-one map $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Notation. $\begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$.

Examples. $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

Proposition. The number of all permutations on the set $\{1, 2, \ldots, n\}$ is $n! = 1 \cdot 2 \cdot \ldots \cdot n$.

[We have n ways to choose $\sigma(1)$, for any choice of $\sigma(1)$ we have $n - 1$ ways to choose $\sigma(2)$, and so on.]

Given two permutations σ_1 and σ_2 on $\{1, 2, \ldots, n\}$, their *composition* $\sigma_1\sigma_2$, $\sigma_1\sigma_2(k) = \sigma_1(\sigma_2(k))$, is also a permutation on $\{1, 2, \ldots, n\}$.

A permutation that exchanges two elements and preserves the others is called a **transposition**.

Theorem Any permutation σ can be decomposed into the composition of several transpositions: $\sigma = \tau_1\tau_2 \ldots \tau_m$.

This decomposition is not unique. Even the number m of transpositions is not uniquely determined. However the parity of m is uniquely determined by the permutation σ. It is called the *parity* of σ.

Example. There are 6 permutations on $\{1, 2, 3\}$:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

σ_4, σ_5, and σ_6 are transpositions, hence they are odd. σ_1 is the identity map, hence it is even.
$\sigma_2 = \sigma_5\sigma_6$ and $\sigma_3 = \sigma_6\sigma_5$, hence σ_2 and σ_3 are even.

Definition. Given an $n \times n$ matrix $A = (a_{ij})$, the **determinant** of A is defined by

$$\det A = \sum_{\sigma} \mathrm{sgn}(\sigma)\, a_{1\sigma(1)}\, a_{2\sigma(2)} \dots a_{n\sigma(n)},$$

where σ runs over all permutations on $\{1, 2, \dots, n\}$. Here $\mathrm{sgn}(\sigma) = 1$ if the permutation σ is even and $\mathrm{sgn}(\sigma) = -1$ if σ is odd.

Properties of determinants

- If one row of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ rc_1 & rc_2 & rc_3 \end{vmatrix} = r \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Suppose that matrices A, B, C are identical except for the ith row and the ith row of C is the sum of the ith rows of A and B.

  Then $\det A = \det B + \det C$.

$$\begin{vmatrix} a_1+a_1' & a_2+a_2' & a_3+a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & a_2' & a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

• Interchanging two rows of a matrix changes the sign of its determinant.

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

• Suppose that a matrix B is obtained from a matrix A by permuting its rows.

Then $\det B = \det A$ if the permutation is even and $\det B = -\det A$ if the permutation is odd.

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- If a matrix A has two identical rows then $\det A = 0$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

- If a matrix A has two rows proportional then $\det A = 0$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ ra_1 & ra_2 & ra_3 \end{vmatrix} = r \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

- Adding a scalar multiple of one row to another does not change the determinant of a matrix.

$$\begin{vmatrix} a_1 + rb_1 & a_2 + rb_2 & a_3 + rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} rb_1 & rb_2 & rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Definition. A square matrix is called **upper triangular** if all entries below the main diagonal are zeros.

• The determinant of an upper triangular matrix is equal to the product of its diagonal entries.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

• If $A = \mathrm{diag}(d_1, d_2, \dots, d_n)$ then $\det A = d_1 d_2 \dots d_n$. In particular, $\det I = 1$.

Determinant of the transpose

- If A is a square matrix then $\det A^T = \det A$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Columns vs. rows

- If one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar.
- Interchanging two columns of a matrix changes the sign of its determinant.
- If a matrix A has two columns proportional then $\det A = 0$.
- Adding a scalar multiple of one column to another does not change the determinant of a matrix.

Submatrices

Definition. Given a matrix A, a $k \times k$ **submatrix** of A is a matrix obtained by specifying k columns and k rows of A and deleting the other columns and rows.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 10 & 20 & 30 & 40 \\ 3 & 5 & 7 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} * & 2 & * & 4 \\ * & * & * & * \\ * & 5 & * & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 4 \\ 5 & 9 \end{pmatrix}$$

If A is an $n \times n$ matrix then A_{ij} denote the $(n-1)\times(n-1)$ submatrix obtained by deleting the ith row and the jth column.

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$.

$$A_{11} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, A_{12} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}, A_{13} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix}, A_{22} = \begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix}, A_{23} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix},$$

$$A_{31} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, A_{32} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, A_{33} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}.$$

Row and column expansions

Theorem Let A be an $n \times n$ matrix. Then for any $1 \leq k, m \leq n$ we have that

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det A_{kj},$$

(*expansion by* k*th row*)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det A_{im}.$$

(*expansion by* m*th column*)

Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$.

Expansion by the 1st row:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -5.$$

Expansion by the 2nd row:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -1\begin{vmatrix} -2 & 0 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = -5.$$

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$.

Expansion by the 2nd column:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -(-2) \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} - 3 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = -5.$$

Expansion by the 3rd column:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -1 \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = -5.$$