Math 304–504

Linear Algebra

Lecture 9:
Evaluation of determinants.

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ij})_{1 \le i,j \le n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A = 0$ if and only if the matrix A is singular.

Explicit definition in low dimensions

Definition. $\det(a) = a$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+ : \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$- : \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Properties of determinants

Determinants and elementary row operations:

- if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;
- if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;
- if we interchange two rows of a matrix, the determinant changes its sign.

Properties of determinants

Tests for singularity:

- if a matrix A has a zero row then $\det A = 0$;
- if a matrix A has two identical rows then $\det A = 0$;
- if a matrix has two proportional rows then $\det A = 0$.

Properties of determinants

Special matrices:

- $\det I = 1$;
- the determinant of a diagonal matrix is equal to the product of its diagonal entries;
- the determinant of an upper triangular matrix is equal to the product of its diagonal entries.

Properties of determinants

Determinant of the transpose:

- If A is a square matrix then $\det A^T = \det A$.

Columns vs. rows:

- if one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar;
- adding a scalar multiple of one column to another does not change the determinant;
- interchanging two columns of a matrix changes the sign of its determinant;
- if a matrix A has a zero row or two proportional columns then $\det A = 0$.

Row and column expansions

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1)\times(n-1)$ submatrix obtained by deleting the ith row and the jth column of A.

Theorem For any $1 \le k, m \le n$ we have that

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj},$$

(*expansion by kth row*)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}.$$

(*expansion by mth column*)

Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Expansion by the 1st row:

$$\begin{pmatrix} \boxed{1} & * & * \\ * & 5 & 6 \\ * & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} * & * & \boxed{3} \\ 4 & 5 & * \\ 7 & 8 & * \end{pmatrix}$$

$$\det A = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0.$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Expansion by the 2nd column:

$$\begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ * & \boxed{5} & * \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ 4 & * & 6 \\ * & \boxed{8} & * \end{pmatrix}$$

$$\det A = -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$$= -2(4 \cdot 9 - 6 \cdot 7) + 5(1 \cdot 9 - 3 \cdot 7) - 8(1 \cdot 6 - 3 \cdot 4) = 0.$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Subtract the 1st row from the 2nd row and from the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$$

since the last matrix has two proportional rows.

Another example. $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{pmatrix}.$

Let's do some row reduction.

Add -4 times the 1st row to the 2nd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix}$$

Add -7 times the 1st row to the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

Expand the determinant by the 1st column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix}$$

Thus

$$\det B = \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix} = (-3) \begin{vmatrix} 1 & 2 \\ -6 & -8 \end{vmatrix}$$

$$= (-3)(-2) \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (-3)(-2)(-2) = -12$$

Example. $C = \begin{pmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{pmatrix}$, $\det C = ?$

Expand the determinant by the 3rd column:

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Add -2 times the 2nd row to the 1st row:

$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det C = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \cdot 9 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

Thus

$$\det C = -18 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -18 \cdot 2 = -36$$

Problem. For what values of a will the following system have a unique solution:

$$\begin{cases} x + 2y + z = 1 \\ -x + 4y + 2z = 2 \\ 2x - 2y + az = 3 \end{cases}$$

The system has a unique solution if and only if the coefficient matrix is invertible.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

Add -2 times the 3rd column to the 2nd column:

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2-2a & a \end{vmatrix}$$

Expand the determinant by the 2nd column:

$$\det A = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2-2a & a \end{vmatrix} = -(-2-2a) \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

Hence $\det A = -(-2-2a) \cdot 3 = 6(1+a)$.
Thus A is invertible if and only if $a \neq -1$.

Determinants and matrix multiplication

Theorem Suppose A and B are $n \times n$ matrices.
Then $\boxed{\det(AB) = \det A \cdot \det B.}$

Corollary 1 $\det(AB) = \det(BA)$.

Proof: $\det(AB) = \det A \, \det B = \det B \, \det A = \det(BA)$.

Corollary 2 $\det(A^{-1}) = (\det A)^{-1}$.

Proof: $\det(A^{-1}) \cdot \det A = \det(A^{-1}A) = \det I = 1$.

Corollary 3 If both A and A^{-1} have integer entries then $\det A = \pm 1$.

Proof: If A and A^{-1} have integer entries then $\det A$ and $\det(A^{-1})$ are integers. But $\det(A^{-1}) \det A = 1$.

We know that $\begin{vmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{vmatrix} = 6(1+a)$.

Let $X = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & -0.5 \end{pmatrix}$.

$\det X = 6, \quad \det Y = 3, \quad \det(XY) = 6 \cdot 3 = 18,$

$\det(Y^{-1}) = 1/3, \quad \det(XY^{-1}) = 6/3 = 2,$

$\det(XYX^{-1}) = \det Y = 3, \quad \det(X^{-1}Y^{-1}XY) = 1,$

$\det(2X) = 2^3 \det X = 2^3 \cdot 6 = 48,$

$\det(-3X^{-1}Y) = (-3)^3 \cdot 6^{-1} \cdot 3 = -27/2,$

$\det(X^T) = \det X = 6, \quad \det(Y^T Y) = (\det Y)^2 = 9.$