MATH 304 Linear Algebra

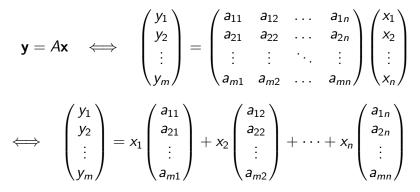
Lecture 16a: Matrix of a linear transformation. Similar matrices.

# Linear transformation

Definition. Given vector spaces 
$$V_1$$
 and  $V_2$ , a mapping  $L: V_1 \rightarrow V_2$  is **linear** if
$$\begin{array}{c} L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}), \\ \hline L(r\mathbf{x}) = rL(\mathbf{x}) \end{array}$$
for any  $\mathbf{x}, \mathbf{y} \in V_1$  and  $r \in \mathbb{R}$ .

#### **Matrix transformations**

**Theorem** Suppose  $L : \mathbb{R}^n \to \mathbb{R}^m$  is a linear map. Then there exists an  $m \times n$  matrix A such that  $L(\mathbf{x}) = A\mathbf{x}$  for all  $\mathbf{x} \in \mathbb{R}^n$ . Columns of A are vectors  $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$ , where  $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$  is the standard basis for  $\mathbb{R}^n$ .



## **Basis and coordinates**

If  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for a vector space V, then any vector  $\mathbf{v} \in V$  has a unique representation

 $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$ 

where  $x_i \in \mathbb{R}$ . The coefficients  $x_1, x_2, \ldots, x_n$  are called the **coordinates** of **v** with respect to the ordered basis  $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ .

The mapping

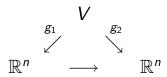
vector  $\mathbf{v} \mapsto its$  coordinates  $(x_1, x_2, \dots, x_n)$ provides a one-to-one correspondence between Vand  $\mathbb{R}^n$ . Besides, this mapping is **linear**.

# **Change of coordinates**

Let V be a vector space.

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be a basis for V and  $g_1 : V \to \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  be another basis for V and  $g_2: V \to \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.



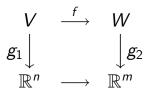
The composition  $g_2 \circ g_1^{-1}$  is a linear mapping of  $\mathbb{R}^n$  to itself. It is represented as  $\mathbf{x} \mapsto U\mathbf{x}$ , where U is an  $n \times n$  matrix. U is called the **transition matrix** from  $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$  to  $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ . Columns of U are coordinates of the vectors

 $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  with respect to the basis  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ .

## Matrix of a linear transformation

Let V, W be vector spaces and  $f: V \to W$  be a linear map. Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be a basis for V and  $g_1: V \to \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.

Let  $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$  be a basis for W and  $g_2 : W \to \mathbb{R}^m$  be the coordinate mapping corresponding to this basis.



The composition  $g_2 \circ f \circ g_1^{-1}$  is a linear mapping of  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . It is represented as  $\mathbf{x} \mapsto A\mathbf{x}$ , where A is an  $m \times n$  matrix.

A is called the **matrix of** f with respect to bases  $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and  $\mathbf{w}_1, \ldots, \mathbf{w}_m$ . Columns of A are coordinates of vectors  $f(\mathbf{v}_1), \ldots, f(\mathbf{v}_n)$  with respect to the basis  $\mathbf{w}_1, \ldots, \mathbf{w}_m$ . *Examples.* •  $D : \mathcal{P}_3 \to \mathcal{P}_2$ , (Dp)(x) = p'(x). Let  $A_D$  be the matrix of D with respect to the bases  $1, x, x^2$  and 1, x. Columns of  $A_D$  are coordinates of polynomials D1, Dx,  $Dx^2$  w.r.t. the basis 1, x.

$$D1 = 0, Dx = 1, Dx^2 = 2x \implies A_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

•  $L: \mathcal{P}_3 \to \mathcal{P}_3$ , (Lp)(x) = p(x+1). Let  $A_L$  be the matrix of L w.r.t. the basis  $1, x, x^2$ .  $L1 = 1, Lx = 1 + x, Lx^2 = (x+1)^2 = 1 + 2x + x^2$ .  $\implies A_L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$  **Problem.** Consider a linear operator *L* on the vector space of  $2 \times 2$  matrices given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Find the matrix of L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Let  $M_L$  denote the desired matrix.

By definition,  $M_L$  is a 4×4 matrix whose columns are coordinates of the matrices  $L(E_1), L(E_2), L(E_3), L(E_4)$ with respect to the basis  $E_1, E_2, E_3, E_4$ .

$$L(E_1) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = 1E_1 + 0E_2 + 3E_3 + 0E_4,$$
  

$$L(E_2) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = 0E_1 + 1E_2 + 0E_3 + 3E_4,$$
  

$$L(E_3) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = 2E_1 + 0E_2 + 4E_3 + 0E_4,$$
  

$$L(E_4) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = 0E_1 + 2E_2 + 0E_3 + 4E_4.$$

It follows that

$$M_L = egin{pmatrix} 1 & 0 & 2 & 0 \ 0 & 1 & 0 & 2 \ 3 & 0 & 4 & 0 \ 0 & 3 & 0 & 4 \end{pmatrix}.$$

Thus the relation

$$\begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

is equivalent to the relation

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

.

**Problem.** Consider a linear operator  $L : \mathbb{R}^2 \to \mathbb{R}^2$ ,

$$L\begin{pmatrix} x\\ y \end{pmatrix} = \begin{pmatrix} 1 & 1\\ 0 & 1 \end{pmatrix} \begin{pmatrix} x\\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis  $\mathbf{v}_1 = (3, 1)$ ,  $\mathbf{v}_2 = (2, 1)$ .

Let *N* be the desired matrix. Columns of *N* are coordinates of the vectors  $L(\mathbf{v}_1)$  and  $L(\mathbf{v}_2)$  w.r.t. the basis  $\mathbf{v}_1, \mathbf{v}_2$ .

$$\begin{split} \mathcal{L}(\mathbf{v}_1) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \mathcal{L}(\mathbf{v}_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \\ \text{Clearly,} \quad \mathcal{L}(\mathbf{v}_2) &= \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2. \end{split}$$

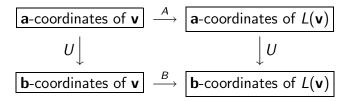
$$L(\mathbf{v}_1) = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \iff \begin{cases} 3\alpha + 2\beta = 4\\ \alpha + \beta = 1 \end{cases} \iff \begin{cases} \alpha = 2\\ \beta = -1 \end{cases}$$
  
Thus  $N = \begin{pmatrix} 2 & 1\\ -1 & 0 \end{pmatrix}$ .

# Change of basis for a linear operator

Let  $L: V \to V$  be a linear operator on a vector space V.

Let A be the matrix of L relative to a basis  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$  for V.

Let U be the transition matrix from the basis  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  to  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ .



It follows that  $UA\mathbf{x} = BU\mathbf{x}$  for all  $\mathbf{x} \in \mathbb{R}^n \implies UA = BU$ . Then  $A = U^{-1}BU$  and  $B = UAU^{-1}$ . **Problem.** Consider a linear operator  $L : \mathbb{R}^2 \to \mathbb{R}^2$ ,  $L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$ 

Find the matrix of L with respect to the basis  $\mathbf{v}_1 = (3, 1)$ ,  $\mathbf{v}_2 = (2, 1)$ .

Let *S* be the matrix of *L* with respect to the standard basis, *N* be the matrix of *L* with respect to the basis  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and *U* be the transition matrix from  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  to  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ . Then  $N = U^{-1}SU$ .

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$
$$N = U^{-1}SU = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

# Similarity

Definition. An  $n \times n$  matrix B is said to be similar to an  $n \times n$  matrix A if  $B = S^{-1}AS$  for some nonsingular  $n \times n$  matrix S.

*Remark.* Two  $n \times n$  matrices are similar if and only if they represent the same linear operator on  $\mathbb{R}^n$  with respect to different bases.

**Theorem** Similarity is an *equivalence relation*, which means that

(i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Theorem Similarity is an equivalence relation, i.e.,
(i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Proof: (i) 
$$A = I^{-1}AI$$
.  
(ii) If  $B = S^{-1}AS$  then  $A = SBS^{-1} = (S^{-1})^{-1}BS^{-1}$   
 $= S_1^{-1}BS_1$ , where  $S_1 = S^{-1}$ .  
(iii) If  $A = S^{-1}BS$  and  $B = T^{-1}CT$  then  
 $A = S^{-1}(T^{-1}CT)S = (S^{-1}T^{-1})C(TS) = (TS)^{-1}C(TS)$   
 $= S_2^{-1}CS_2$ , where  $S_2 = TS$ .

**Theorem** If A and B are similar matrices then they have the same (i) determinant, (ii) trace = the sum of diagonal entries, (iii) rank, and (iv) nullity.