MATH 304

Linear Algebra

Lecture 6:
Diagonal matrices.
Inverse matrix.

Matrices

Definition. An **m-by-n matrix** is a rectangular array of numbers that has m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Notation: $A = (a_{ij})_{1 \le i \le n,\, 1 \le j \le m}$ or simply $A = (a_{ij})$ if the dimensions are known.

Matrix algebra: linear operations

Addition: two matrices of the same dimensions can be added by adding their corresponding entries.

Scalar multiplication: to multiply a matrix A by a scalar r, one multiplies each entry of A by r.

Zero matrix O: all entries are zeros.

Negative: $-A$ is defined as $(-1)A$.

Subtraction: $A - B$ is defined as $A + (-B)$.

As far as the linear operations are concerned, the $m \times n$ matrices can be regarded as mn-dimensional vectors.

Matrix algebra: matrix multiplication

The product of matrices A and B is defined if the number of columns in A matches the number of rows in B.

Definition. Let $A = (a_{ik})$ be an $m \times n$ matrix and $B = (b_{kj})$ be an $n \times p$ matrix. The **product** AB is defined to be the $m \times p$ matrix $C = (c_{ij})$ such that $\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$ for all indices i, j.

That is, matrices are multiplied **row by column**.

$$A = \left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ \hline a_{21} & a_{22} & \dots & a_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} \end{array}\right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

$$B = \left(\begin{array}{c|c|c|c} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{array}\right) = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p)$$

$$\implies \quad AB = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{w}_1 & \mathbf{v}_1 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_1 \cdot \mathbf{w}_p \\ \mathbf{v}_2 \cdot \mathbf{w}_1 & \mathbf{v}_2 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_2 \cdot \mathbf{w}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_m \cdot \mathbf{w}_1 & \mathbf{v}_m \cdot \mathbf{w}_2 & \dots & \mathbf{v}_m \cdot \mathbf{w}_p \end{pmatrix}$$

Diagonal matrices

If $A = (a_{ij})$ is a square matrix, then the entries a_{ii} are called **diagonal entries**. A square matrix is called **diagonal** if all non-diagonal entries are zeros.

Example. $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, denoted $\mathrm{diag}(7, 1, 2)$.

Let $A = \mathrm{diag}(s_1, s_2, \dots, s_n)$, $B = \mathrm{diag}(t_1, t_2, \dots, t_n)$.

Then $A + B = \mathrm{diag}(s_1 + t_1, s_2 + t_2, \dots, s_n + t_n)$,

$$rA = \mathrm{diag}(rs_1, rs_2, \dots, rs_n).$$

Example.

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Theorem Let $A = \mathrm{diag}(s_1, s_2, \ldots, s_n)$, $B = \mathrm{diag}(t_1, t_2, \ldots, t_n)$.

Then $A + B = \mathrm{diag}(s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$,

$$rA = \mathrm{diag}(rs_1, rs_2, \ldots, rs_n).$$

$$AB = \mathrm{diag}(s_1 t_1, s_2 t_2, \ldots, s_n t_n).$$

In particular, diagonal matrices always commute (i.e., $AB = BA$).

Example.

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 7a_{11} & 7a_{12} & 7a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{pmatrix}$$

Theorem Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_m)$ and A be an $m \times n$ matrix. Then the matrix DA is obtained from A by multiplying the ith row by d_i for $i = 1, 2, \ldots, m$:

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \quad \Longrightarrow \quad DA = \begin{pmatrix} d_1\mathbf{v}_1 \\ d_2\mathbf{v}_2 \\ \vdots \\ d_m\mathbf{v}_m \end{pmatrix}$$

Example.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 7a_{11} & a_{12} & 2a_{13} \\ 7a_{21} & a_{22} & 2a_{23} \\ 7a_{31} & a_{32} & 2a_{33} \end{pmatrix}$$

Theorem Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and A be an $m \times n$ matrix. Then the matrix AD is obtained from A by multiplying the ith column by d_i for $i = 1, 2, \ldots, n$:

$$\begin{aligned} A &= (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n) \\ \Longrightarrow \quad AD &= (d_1\mathbf{w}_1, d_2\mathbf{w}_2, \ldots, d_n\mathbf{w}_n) \end{aligned}$$

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1. The $n \times n$ identity matrix is denoted I_n or simply I.

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$

Theorem. Let A be an arbitrary $m \times n$ matrix. Then $I_m A = A I_n = A$.

Inverse matrix

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. We can **add**, **subtract**, and **multiply** elements of $\mathcal{M}_n(\mathbb{R})$. What about **division**?

Definition. Let $A \in \mathcal{M}_n(\mathbb{R})$. Suppose there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

Then the matrix A is called **invertible** and B is called the **inverse** of A (denoted A^{-1}).

A non-invertible square matrix is called **singular**.

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Examples

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $A^{-1} = B$, $B^{-1} = A$, and $C^{-1} = C$.

Inverting diagonal matrices

Theorem A diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

$$\begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n \end{pmatrix}^{-1} = \begin{pmatrix} d_1^{-1} & 0 & \ldots & 0 \\ 0 & d_2^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n^{-1} \end{pmatrix}$$

Inverting diagonal matrices

Theorem A diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

Proof: If all $d_i \neq 0$ then, clearly,

$$\mathrm{diag}(d_1, \ldots, d_n)\, \mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1}) = \mathrm{diag}(1, \ldots, 1) = I,$$

$$\mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})\, \mathrm{diag}(d_1, \ldots, d_n) = \mathrm{diag}(1, \ldots, 1) = I.$$

Now suppose that $d_i = 0$ for some i. Then for any $n \times n$ matrix B the ith row of the matrix DB is a zero row. Hence $DB \neq I$.

Inverting 2×2 matrices

Definition. The **determinant** of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$.

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$.

If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$. If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof: Let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = BA = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad - bc)I_2.$$

In the case $\det A \neq 0$, we have $A^{-1} = (\det A)^{-1}B$.
In the case $\det A = 0$, the matrix A is not invertible as otherwise $AB = O \implies A^{-1}(AB) = A^{-1}O = O$
$\implies (A^{-1}A)B = O \implies I_2B = O \implies B = O$
$\implies A = O$, but the zero matrix is singular.