MATH 304 Linear Algebra Lecture 21: General linear equations. Matrix transformations. Matrix of a linear transformation.

Linear transformation

Definition. Given vector spaces V_1 and V_2 , a mapping $L: V_1 \rightarrow V_2$ is **linear** if $\begin{array}{c}
L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}), \\
\hline
L(r\mathbf{x}) = rL(\mathbf{x})
\end{array}$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Basic properties of linear mappings:

•
$$L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$$

for all $k \ge 1$, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V_1$, and $r_1, \ldots, r_k \in \mathbb{R}$.

• $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2 , respectively.

•
$$L(-\mathbf{v}) = -L(\mathbf{v})$$
 for any $\mathbf{v} \in V_1$.

Range and kernel

Let V, W be vector spaces and $L: V \rightarrow W$ be a linear mapping.

Definition. The range (or image) of *L* is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of *L* is denoted L(V).

The **kernel** of *L*, denoted ker *L*, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem (i) The range of L is a subspace of W. (ii) The kernel of L is a subspace of V.

General linear equations

Definition. A **linear equation** is an equation of the form

$$L(\mathbf{x}) = \mathbf{b}$$
,

where $L: V \to W$ is a linear mapping, **b** is a given vector from W, and **x** is an unknown vector from V.

The range of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The kernel of *L* is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable then the general solution is

$$\mathbf{x}_0 + t_1 \mathbf{v}_1 + \cdots + t_k \mathbf{v}_k$$
,

where \mathbf{x}_0 is a particular solution, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a basis for the kernel of L, and t_1, \ldots, t_k are arbitrary scalars.

Example.
$$\begin{cases} x + y + z = 4, \\ x + 2y = 3. \end{cases}$$

 $L : \mathbb{R}^3 \to \mathbb{R}^2, \quad L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$
Linear equation: $L(\mathbf{x}) = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$
 $\begin{pmatrix} 1 & 1 & 1 & | & 4 \\ 1 & 2 & 0 & | & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & | & 4 \\ 0 & 1 & -1 & | & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & | & 5 \\ 0 & 1 & -1 & | & -1 \end{pmatrix}$
 $\begin{cases} x + 2z = 5 \\ y - z = -1 \end{cases} \iff \begin{cases} x = 5 - 2z \\ y = -1 + z \end{cases}$
 $(x, y, z) = (5 - 2t, -1 + t, t) = (5, -1, 0) + t(-2, 1, 1).$

Example. $u''(x) + u(x) = e^{2x}$.

Linear operator $L: C^2(\mathbb{R}) \to C(\mathbb{R}), Lu = u'' + u$. Linear equation: Lu = b, where $b(x) = e^{2x}$.

It can be shown that the range of L is the entire space $C(\mathbb{R})$ while the kernel of L is spanned by the functions $\sin x$ and $\cos x$.

Particular solution: $u_0 = \frac{1}{5}e^{2x}$.

Thus the general solution is

$$u(x)=\tfrac{1}{5}e^{2x}+t_1\sin x+t_2\cos x.$$

Matrix transformations

Any $m \times n$ matrix A gives rise to a transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $L(\mathbf{x}) \in \mathbb{R}^m$ are regarded as column vectors. This transformation is **linear**.

Example.
$$L\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1 & 0 & 2\\3 & 4 & 7\\0 & 5 & 8\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis for \mathbb{R}^3 . We have that $L(\mathbf{e}_1) = (1, 3, 0)$, $L(\mathbf{e}_2) = (0, 4, 5)$, $L(\mathbf{e}_3) = (2, 7, 8)$. Thus $L(\mathbf{e}_1)$, $L(\mathbf{e}_2)$, $L(\mathbf{e}_3)$ are columns of the matrix.

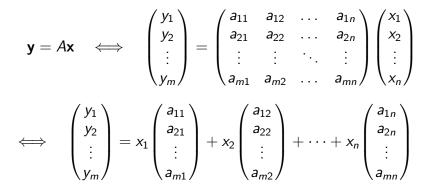
Problem. Find a linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(\mathbf{e}_1) = (1, 1)$, $L(\mathbf{e}_2) = (0, -2)$, $L(\mathbf{e}_3) = (3, 0)$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for \mathbb{R}^3 .

$$L(x, y, z) = L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3)$$

= $xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3)$
= $x(1, 1) + y(0, -2) + z(3, 0) = (x + 3z, x - 2y)$
 $L(x, y, z) = \begin{pmatrix} x + 3z \\ x - 2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Columns of the matrix are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$.

Theorem Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is the standard basis for \mathbb{R}^n .

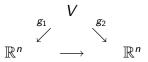


Change of coordinates (revisited)

Let V be a vector space.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for V and $g_2: V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

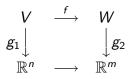


The composition $g_2 \circ g_1^{-1}$ is a **linear** mapping of \mathbb{R}^n to itself. It is represented as $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix. U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Matrix of a linear transformation

Let V, W be vector spaces and $f: V \to W$ be a linear map. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1: V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ be a basis for W and $g_2 : W \to \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.



The composition $g_2 \circ f \circ g_1^{-1}$ is a **linear** mapping of \mathbb{R}^n to \mathbb{R}^m . It is represented as $\mathbf{x} \mapsto A\mathbf{x}$, where A is an $m \times n$ matrix. A is called the **matrix of** f with respect to bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \ldots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \ldots, \mathbf{w}_m$. *Examples.* • $D : \mathcal{P}_3 \to \mathcal{P}_2$, (Dp)(x) = p'(x). Let A_D be the matrix of D with respect to the bases $1, x, x^2$ and 1, x. Columns of A_D are coordinates of polynomials D1, Dx, Dx^2 w.r.t. the basis 1, x.

$$D1 = 0, Dx = 1, Dx^2 = 2x \implies A_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

• $L: \mathcal{P}_3 \to \mathcal{P}_3$, (Lp)(x) = p(x+1). Let A_L be the matrix of L w.r.t. the basis $1, x, x^2$. $L1 = 1, Lx = 1 + x, Lx^2 = (x+1)^2 = 1 + 2x + x^2$. $\implies A_L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$