MATH 304

Linear Algebra

Lecture 33: Diagonalization (continued).

Diagonalization

Let L be a linear operator on a finite-dimensional vector space V. Then the following conditions are equivalent:

- the matrix of L with respect to some basis is diagonal;
- there exists a basis for V formed by eigenvectors of L.

The operator L is **diagonalizable** if it satisfies these conditions.

Let A be an $n \times n$ matrix. Then the following conditions are equivalent:

- A is the matrix of a diagonalizable operator;
- A is similar to a diagonal matrix, i.e., it is represented as $A = UBU^{-1}$, where the matrix B is diagonal;
- there exists a basis for $\mathbb{R}^n$ formed by eigenvectors of A.

The matrix A is **diagonalizable** if it satisfies these conditions. Otherwise A is called **defective**.

Theorem 1 If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of a linear operator L associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Theorem 2 Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of a linear operator L. For any $1 \leq i \leq k$ let S_i be a basis for the eigenspace associated with the eigenvalue λ_i. Then the union $S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent set.

Corollary Let A be an $n \times n$ matrix such that the characteristic equation $\det(A - \lambda I) = 0$ has n distinct real roots. Then

(i) there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of A;

(ii) all eigenspaces of A are one-dimensional.

There are **two obstructions** to existence of a basis consisting of eigenvectors. They are illustrated by the following examples.

Example 1. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\det(A - \lambda I) = (\lambda - 1)^2$. Hence $\lambda = 1$ is the only eigenvalue. The associated eigenspace is the line $t(1, 0)$.

Example 2. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = \lambda^2 + 1$.

$\implies$ no real eigenvalues or eigenvectors

(However there are *complex* eigenvalues/eigenvectors.)

Matrix polynomials

Definition. Given an n-by-n matrix A, we let

$$A^2 = AA, \quad A^3 = AAA, \; \dots, \quad A^k = \underbrace{AA\dots A}_{k \text{ times}}, \; \dots$$

Also, let $A^1 = A$ and $A^0 = I_n$.

Associativity of matrix multiplication implies that all powers A^k are well defined and $A^j A^k = A^{j+k}$ for all $j, k \geq 0$. In particular, all powers of A commute.

Definition. For any polynomial

$$p(x) = c_0 x^m + c_1 x^{m-1} + \cdots + c_{m-1} x + c_m,$$

let $p(A) = c_0 A^m + c_1 A^{m-1} + \cdots + c_{m-1} A + c_m I_n$.

Theorem If $A = \mathrm{diag}(a_1, a_2, \dots, a_n)$, then $p(A) = \mathrm{diag}\big(p(a_1), p(a_2), \dots, p(a_n)\big)$.

Let A be an n-by-n matrix and suppose there exists a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ for $\mathbb{R}^n$ consisting of eigenvectors of A. That is, $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$, where $\lambda_k \in \mathbb{R}$.
Then $A = UBU^{-1}$, where $B = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and U is a transition matrix whose columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

$A^2 = UBU^{-1}UBU^{-1} = UB^2U^{-1}$,
$A^3 = A^2A = UB^2U^{-1}UBU^{-1} = UB^3U^{-1}$.
Likewise, $A^n = UB^nU^{-1}$ for any $n \geq 1$.

$I + 2A - 3A^2 = UIU^{-1} + 2UBU^{-1} - 3UB^2U^{-1} =$
$= U(I + 2B - 3B^2)U^{-1}$.
Likewise, $p(A) = Up(B)U^{-1}$ for any polynomial $p(x)$.

Problem. Diagonalize the matrix $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$.

Characteristic equation of A: $\begin{vmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = 0$.

$(4-\lambda)(1-\lambda) = 0 \implies \lambda_1 = 4,\ \lambda_2 = 1$.

Associated eigenvectors: $\mathbf{v}_1 = (1,0)$, $\mathbf{v}_2 = (-1,1)$.

Thus $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Problem. Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find a matrix C such that $C^2 = A$.

We know that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Suppose that $D^2 = B$ for some matrix D. Let $C = UDU^{-1}$.
Then $C^2 = UDU^{-1}UDU^{-1} = UD^2U^{-1} = UBU^{-1} = A$.

We can take $D = \begin{pmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

Initial value problem for a system of linear ODEs:

$$\begin{cases} \frac{dx}{dt} = 4x + 3y, \\ \frac{dy}{dt} = y, \end{cases} \qquad x(0) = 1, \;\; y(0) = 1.$$

The system can be rewritten in vector form:

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v}, \quad \text{where} \;\; A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}, \;\; \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Matrix A is diagonalizable: $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be coordinates of the vector $\mathbf{v}$ relative to the basis $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (-1, 1)$ of eigenvectors of A. Then $\mathbf{v} = U\mathbf{w} \implies \mathbf{w} = U^{-1}\mathbf{v}$.

It follows that

$$\frac{d\mathbf{w}}{dt} = \frac{d}{dt}(U^{-1}\mathbf{v}) = U^{-1}\frac{d\mathbf{v}}{dt} = U^{-1}A\mathbf{v} = U^{-1}AU\mathbf{w}.$$

Hence $\quad \dfrac{d\mathbf{w}}{dt} = B\mathbf{w} \quad \iff \quad \begin{cases} \frac{dw_1}{dt} = 4w_1, \\ \frac{dw_2}{dt} = w_2. \end{cases}$

General solution: $w_1(t) = c_1 e^{4t}$, $w_2(t) = c_2 e^{t}$, where $c_1, c_2 \in \mathbb{R}$.

Initial condition:

$$\mathbf{w}(0) = U^{-1}\mathbf{v}(0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus $w_1(t) = 2e^{4t}$, $w_2(t) = e^{t}$. Then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = U\mathbf{w}(t) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2e^{4t} \\ e^{t} \end{pmatrix} = \begin{pmatrix} 2e^{4t} - e^{t} \\ e^{t} \end{pmatrix}.$$