MATH 304–505 Spring 2011

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Find a quadratic polynomial p(x) such that p(-1) = p(3) = 6 and p'(2) = p(1).

Problem 2 (20 pts.) Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on \mathbb{R}^3 such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.

- (i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for \mathbb{R}^3 .
- (ii) Find the matrix of the operator L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
- (iii) Find the matrix of the operator L relative to the standard basis.

Problem 3 (20 pts.) Let
$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}$$
.

- (i) Evaluate the determinant of the matrix A.
- (ii) Find the inverse matrix A^{-1} .

Problem 4 (25 pts.) Let
$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
.

- (i) Find all eigenvalues of the matrix B.
- (ii) Find a basis for \mathbb{R}^3 consisting of eigenvectors of B.
- (iii) Find an orthonormal basis for \mathbb{R}^3 consisting of eigenvectors of B.
- (iv) Find a diagonal matrix X and an invertible matrix U such that $B = UXU^{-1}$.

Problem 5 (20 pts.) Let *V* be a subspace of \mathbb{R}^4 spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (2, 0, -1, 1)$, and $\mathbf{x}_3 = (0, 1, 1, 0)$.

- (i) Find the distance from the point $\mathbf{y} = (0, 0, 0, 4)$ to the subspace V.
- (ii) Find the distance from the point y to the orthogonal complement V^{\perp} .

Bonus Problem 6 (15 pts.) (i) Find a matrix exponential $\exp(tC)$, where $C = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ and $t \in \mathbb{R}$.

(ii) Solve a system of differential equations

$$\begin{cases} \frac{dx}{dt} = 3x + y, \\ \frac{dy}{dt} = 3y \end{cases}$$

subject to the initial conditions x(0) = y(0) = 1.

Bonus Problem 7 (15 pts.) Consider a linear operator $K: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$K(\mathbf{x}) = D\mathbf{x}$$
, where $D = \frac{1}{9} \begin{pmatrix} -4 & 7 & 4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}$.

The operator K is a rotation about an axis.

- (i) Find the axis of rotation.
- (ii) Find the angle of rotation.