MATH 304

Linear Algebra

Lecture 8:
Vector spaces.
Subspaces.

Linear operations on vectors

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be n-dimensional vectors, and $r \in \mathbb{R}$ be a scalar.

Vector sum: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

Scalar multiple: $r\mathbf{x} = (rx_1, rx_2, \ldots, rx_n)$

Zero vector: $\mathbf{0} = (0, 0, \ldots, 0)$

Negative of a vector: $-\mathbf{y} = (-y_1, -y_2, \ldots, -y_n)$

Vector difference:
$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}) = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$

Properties of linear operations

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$

$$\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$$

$$\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$$

$$r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$$

$$(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$$

$$(rs)\mathbf{x} = r(s\mathbf{x})$$

$$1\mathbf{x} = \mathbf{x}$$

$$0\mathbf{x} = \mathbf{0}$$

$$(-1)\mathbf{x} = -\mathbf{x}$$

Linear operations on matrices

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices, and $r \in \mathbb{R}$ be a scalar.

Matrix sum: $A + B = (a_{ij} + b_{ij})_{1 \le i \le m,\ 1 \le j \le n}$

Scalar multiple: $rA = (ra_{ij})_{1 \le i \le m,\ 1 \le j \le n}$

Zero matrix O: all entries are zeros

Negative of a matrix: $-A = (-a_{ij})_{1 \le i \le m,\ 1 \le j \le n}$

Matrix difference: $A - B = (a_{ij} - b_{ij})_{1 \le i \le m,\ 1 \le j \le n}$

As far as the linear operations are concerned, the $m \times n$ matrices have the same properties as mn-dimensional vectors.

Abstract vector space: informal description

Vector space = *linear space* = a set V of objects (called *vectors*) that can be added and scaled.

That is, for any $\mathbf{u}, \mathbf{v} \in V$ and $r \in \mathbb{R}$ expressions

$$\boxed{\mathbf{u} + \mathbf{v}} \text{ and } \boxed{r\mathbf{u}}$$

should make sense.

Certain restrictions apply. For instance,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u},$$
$$2\mathbf{u} + 3\mathbf{u} = 5\mathbf{u}.$$

That is, addition and scalar multiplication in V should be like those of n-dimensional vectors.

Abstract vector space: definition

Vector space is a set V equipped with two operations $\alpha : V \times V \to V$ and $\mu : \mathbb{R} \times V \to V$ that have certain properties (listed below).

The operation α is called *addition*. For any $\mathbf{u}, \mathbf{v} \in V$, the element $\alpha(\mathbf{u}, \mathbf{v})$ is denoted $\mathbf{u} + \mathbf{v}$.

The operation μ is called *scalar multiplication*. For any $r \in \mathbb{R}$ and $\mathbf{u} \in V$, the element $\mu(r, \mathbf{u})$ is denoted $r\mathbf{u}$.

Properties of addition and scalar multiplication (brief)

A1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

A2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

A3. $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$

A4. $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$

A5. $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$

A6. $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$

A7. $(rs)\mathbf{a} = r(s\mathbf{a})$

A8. $1\mathbf{a} = \mathbf{a}$

Properties of addition and scalar multiplication (detailed)

A1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ for all $\mathbf{a}, \mathbf{b} \in V$.

A2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$.

A3. There exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ for all $\mathbf{a} \in V$.

A4. For any $\mathbf{a} \in V$ there exists an element of V, denoted $-\mathbf{a}$, such that $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$.

A5. $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$ for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in V$.

A6. $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$ for all $r, s \in \mathbb{R}$ and $\mathbf{a} \in V$.

A7. $(rs)\mathbf{a} = r(s\mathbf{a})$ for all $r, s \in \mathbb{R}$ and $\mathbf{a} \in V$.

A8. $1\mathbf{a} = \mathbf{a}$ for all $\mathbf{a} \in V$.

- Associativity of addition implies that a multiple sum $\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_k$ is well defined for any $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$.

- **Subtraction** in V is defined as usual: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

- Addition and scalar multiplication are called **linear operations**.

Given $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$ and $r_1, r_2, \ldots, r_k \in \mathbb{R}$,

$$\boxed{r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k}$$

is called a **linear combination** of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$.

Examples of vector spaces

In most examples, addition and scalar multiplication are natural operations so that properties A1–A8 are easy to verify.

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m \times n$ matrices with real entries
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \dots)$, $x_i \in \mathbb{R}$

For any $\mathbf{x} = (x_1, x_2, \dots)$, $\mathbf{y} = (y_1, y_2, \dots) \in \mathbb{R}^\infty$ and $r \in \mathbb{R}$
let $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots)$, $r\mathbf{x} = (rx_1, rx_2, \dots)$.
Then $\mathbf{0} = (0, 0, \dots)$ and $-\mathbf{x} = (-x_1, -x_2, \dots)$.

- $\{\mathbf{0}\}$: the trivial vector space

$\mathbf{0} + \mathbf{0} = \mathbf{0}$, $r\mathbf{0} = \mathbf{0}$, $-\mathbf{0} = \mathbf{0}$.

Functional vector spaces

- $F(\mathbb{R})$: the set of all functions $f : \mathbb{R} \to \mathbb{R}$

Given functions $f, g \in F(\mathbb{R})$ and a scalar $r \in \mathbb{R}$, let $(f+g)(x) = f(x) + g(x)$ and $(rf)(x) = rf(x)$ for all $x \in \mathbb{R}$. Zero vector: $o(x) = 0$. Negative: $(-f)(x) = -f(x)$.

- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

Linear operations are inherited from $F(\mathbb{R})$. We only need to check that $f, g \in C(\mathbb{R}) \implies f+g, rf \in C(\mathbb{R})$, the zero function is continuous, and $f \in C(\mathbb{R}) \implies -f \in C(\mathbb{R})$.

- $C^1(\mathbb{R})$: all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$
- $C^\infty(\mathbb{R})$: all smooth functions $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

Some general observations

- The zero vector is unique.

If $\mathbf{z}_1$ and $\mathbf{z}_2$ are zeros then $\mathbf{z}_1 = \mathbf{z}_1 + \mathbf{z}_2 = \mathbf{z}_2$.

- For any $\mathbf{a} \in V$, the negative $-\mathbf{a}$ is unique.

Suppose $\mathbf{b}$ and $\mathbf{b}'$ are negatives of $\mathbf{a}$. Then
$\mathbf{b}' = \mathbf{b}' + \mathbf{0} = \mathbf{b}' + (\mathbf{a} + \mathbf{b}) = (\mathbf{b}' + \mathbf{a}) + \mathbf{b} = \mathbf{0} + \mathbf{b} = \mathbf{b}$.

- $0\mathbf{a} = \mathbf{0}$ for any $\mathbf{a} \in V$.

Indeed, $0\mathbf{a} + \mathbf{a} = 0\mathbf{a} + 1\mathbf{a} = (0 + 1)\mathbf{a} = 1\mathbf{a} = \mathbf{a}$.
Then $0\mathbf{a} + \mathbf{a} = \mathbf{a} \implies 0\mathbf{a} + \mathbf{a} - \mathbf{a} = \mathbf{a} - \mathbf{a} \implies 0\mathbf{a} = \mathbf{0}$.

- $(-1)\mathbf{a} = -\mathbf{a}$ for any $\mathbf{a} \in V$.

Indeed, $\mathbf{a} + (-1)\mathbf{a} = (-1)\mathbf{a} + \mathbf{a} = (-1)\mathbf{a} + 1\mathbf{a} = (-1 + 1)\mathbf{a}$
$= 0\mathbf{a} = \mathbf{0}$.

Counterexample: dumb scaling

Consider the set $V = \mathbb{R}^n$ with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{a} = \mathbf{0}} \quad \text{for any } \mathbf{a} \in \mathbb{R}^n \text{ and } r \in \mathbb{R}.$$

Properties A1–A4 hold because they do not involve scalar multiplication.

$$\begin{array}{lll}
\text{A5. } r \odot (\mathbf{a} + \mathbf{b}) = r \odot \mathbf{a} + r \odot \mathbf{b} & \iff & \mathbf{0} = \mathbf{0} + \mathbf{0} \\
\text{A6. } (r + s) \odot \mathbf{a} = r \odot \mathbf{a} + s \odot \mathbf{a} & \iff & \mathbf{0} = \mathbf{0} + \mathbf{0} \\
\text{A7. } (rs) \odot \mathbf{a} = r \odot (s \odot \mathbf{a}) & \iff & \mathbf{0} = \mathbf{0} \\
\text{A8. } 1 \odot \mathbf{a} = \mathbf{a} & \iff & \mathbf{0} = \mathbf{a}
\end{array}$$

A8 is the only property that fails. As a consequence, property A8 does not follow from properties A1–A7.

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Examples.

- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

$C(\mathbb{R})$ is a subspace of $F(\mathbb{R})$.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_k x^k$
- $\mathcal{P}_n$: polynomials of degree less than n

$\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

Subspaces of vector spaces

Counterexamples.

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathbb{Q}^n$: vectors with rational coordinates

$\mathbb{Q}^n$ is not a subspace of $\mathbb{R}^n$.

$\sqrt{2}(1,1,\dots,1) \notin \mathbb{Q}^n \implies \mathbb{Q}^n$ is not a vector space (scaling is not well defined).

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$
- P_n^*: polynomials of degree n $(n > 0)$

P_n^* is not a subspace of $\mathcal{P}$.

$-x^n + (x^n + 1) = 1 \notin P_n^* \implies P_n^*$ is not a vector space (addition is not well defined).

If S is a subset of a vector space V then S inherits from V addition and scalar multiplication. However S need not be closed under these operations.

Proposition A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

Proof: "only if" is obvious.
"if": properties like associative, commutative, or distributive law hold for S because they hold for V. We only need to verify properties A3 and A4. Take any $\mathbf{x} \in S$ (note that S is nonempty). Then $\mathbf{0} = 0\mathbf{x} \in S$. Also, $-\mathbf{x} = (-1)\mathbf{x} \in S$.

Example. $V = \mathbb{R}^2$.

- The line $x - y = 0$ is a subspace of $\mathbb{R}^2$.

The line consists of all vectors of the form (t, t), $t \in \mathbb{R}$.

$(t, t) + (s, s) = (t + s, t + s) \implies$ closed under addition
$r(t, t) = (rt, rt) \implies$ closed under scaling

- The parabola $y = x^2$ is not a subspace of $\mathbb{R}^2$.

It is enough to find one explicit counterexample.

Counterexample 1: $(1, 1) + (-1, 1) = (0, 2)$.

$(1, 1)$ and $(-1, 1)$ lie on the parabola while $(0, 2)$ does not
$\implies$ not closed under addition

Counterexample 2: $2(1, 1) = (2, 2)$.

$(1, 1)$ lies on the parabola while $(2, 2)$ does not
$\implies$ not closed under scaling

Example. $V = \mathbb{R}^3$.

- The plane $z = 0$ is a subspace of $\mathbb{R}^3$.
- The plane $z = 1$ is not a subspace of $\mathbb{R}^3$.
- The line $t(1, 1, 0)$, $t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ and a subspace of the plane $z = 0$.
- The line $(1, 1, 1) + t(1, -1, 0)$, $t \in \mathbb{R}$ is not a subspace of $\mathbb{R}^3$ as it lies in the plane $x + y + z = 3$, which does not contain $\mathbf{0}$.
- In general, a straight line or a plane in $\mathbb{R}^3$ is a subspace if and only if it passes through the origin.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Any solution $(x_1, x_2, \ldots, x_n)$ is an element of $\mathbb{R}^n$.

Theorem The solution set of the system is a subspace of $\mathbb{R}^n$ if and only if all $b_i = 0$.