

MATH 304  
Linear Algebra

**Lecture 11:**  
**Basis and dimension.**

## Linear independence

*Definition.* Let  $V$  be a vector space. Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$  are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients  $r_1, \dots, r_k \in \mathbb{R}$  are not all equal to zero. Otherwise the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \dots = r_k = 0.$$

An infinite set  $S \subset V$  is **linearly dependent** if there are some linearly dependent vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$ . Otherwise  $S$  is **linearly independent**.

*Remark.* If a set  $S$  (finite or infinite) is linearly independent then any subset of  $S$  is also linearly independent.

**Theorem** Vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$  are linearly dependent if and only if one of them is a linear combination of the other  $k - 1$  vectors.

*Examples of linear independence.*

- Vectors  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$ , and  $\mathbf{e}_3 = (0, 0, 1)$  in  $\mathbb{R}^3$ .

- Matrices  $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,

$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ , and  $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ .

- Polynomials  $1, x, x^2, \dots, x^n, \dots$

## Spanning set

Let  $S$  be a subset of a vector space  $V$ .

*Definition.* The **span** of the set  $S$  is the smallest subspace  $W \subset V$  that contains  $S$ . If  $S$  is not empty then  $W = \text{Span}(S)$  consists of all linear combinations  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$  such that  $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$  and  $r_1, \dots, r_k \in \mathbb{R}$ .

We say that the set  $S$  **spans** the subspace  $W$  or that  $S$  is a **spanning set** for  $W$ .

*Remark.* If  $S_1$  is a spanning set for a vector space  $V$  and  $S_1 \subset S_2 \subset V$ , then  $S_2$  is also a spanning set for  $V$ .

## Basis

*Definition.* Let  $V$  be a vector space. A linearly independent spanning set for  $V$  is called a **basis**.

Suppose that a set  $S \subset V$  is a basis for  $V$ .

“Spanning set” means that any vector  $\mathbf{v} \in V$  can be represented as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are distinct vectors from  $S$  and  $r_1, \dots, r_k \in \mathbb{R}$ . “Linearly independent” implies that the above representation is unique:

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = r'_1\mathbf{v}_1 + r'_2\mathbf{v}_2 + \cdots + r'_k\mathbf{v}_k$$

$$\implies (r_1 - r'_1)\mathbf{v}_1 + (r_2 - r'_2)\mathbf{v}_2 + \cdots + (r_k - r'_k)\mathbf{v}_k = \mathbf{0}$$

$$\implies r_1 - r'_1 = r_2 - r'_2 = \cdots = r_k - r'_k = 0$$

*Examples.* • Standard basis for  $\mathbb{R}^n$ :

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0, 0), \dots, \\ \mathbf{e}_n = (0, 0, 0, \dots, 0, 1).$$

Indeed,  $(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$ .

- Matrices  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a basis for  $\mathcal{M}_{2,2}(\mathbb{R})$ .

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Polynomials  $1, x, x^2, \dots, x^{n-1}$  form a basis for  $\mathcal{P}_n = \{a_0 + a_1x + \dots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$ .

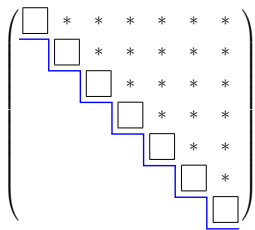
- The infinite set  $\{1, x, x^2, \dots, x^n, \dots\}$  is a basis for  $\mathcal{P}$ , the space of all polynomials.

Let  $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$  and  $r_1, r_2, \dots, r_k \in \mathbb{R}$ .  
The vector equation  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{v}$  is  
equivalent to the matrix equation  $A\mathbf{x} = \mathbf{v}$ , where

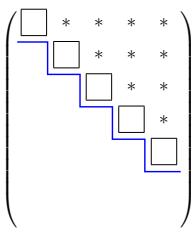
$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

That is,  $A$  is the  $n \times k$  matrix such that vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$   
are consecutive columns of  $A$ .

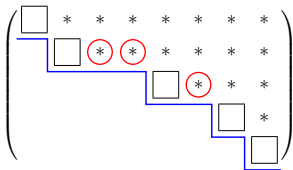
- *Vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  span  $\mathbb{R}^n$  if the row echelon form of  $A$  has no zero rows.*
- *Vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are linearly independent if the row echelon form of  $A$  has a leading entry in each column (no free variables).*



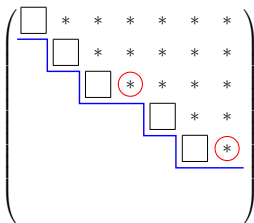
spanning  
linear independence



no spanning  
linear independence



spanning  
no linear independence



no spanning  
no linear independence



## Bases for $\mathbb{R}^n$

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be vectors in  $\mathbb{R}^n$ .

**Theorem 1** If  $k < n$  then the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  do not span  $\mathbb{R}^n$ .

**Theorem 2** If  $k > n$  then the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly dependent.

**Theorem 3** If  $k = n$  then the following conditions are equivalent:

- (i)  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for  $\mathbb{R}^n$ ;
- (ii)  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a spanning set for  $\mathbb{R}^n$ ;
- (iii)  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a linearly independent set.

*Example.* Consider vectors  $\mathbf{v}_1 = (1, -1, 1)$ ,  $\mathbf{v}_2 = (1, 0, 0)$ ,  $\mathbf{v}_3 = (1, 1, 1)$ , and  $\mathbf{v}_4 = (1, 2, 4)$  in  $\mathbb{R}^3$ .

Vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent (as they are not parallel), but they do not span  $\mathbb{R}^3$ .

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-2) = 2 \neq 0.$$

Therefore  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis for  $\mathbb{R}^3$ .

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  span  $\mathbb{R}^3$  (because  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  already span  $\mathbb{R}^3$ ), but they are linearly dependent.

**Problem.** Find a basis for the plane  $x + 2z = 0$  in  $\mathbb{R}^3$ .

The general solution of the equation  $x + 2z = 0$  is

$$\begin{cases} x = -2s \\ y = t \\ z = s \end{cases} \quad (t, s \in \mathbb{R})$$

That is,  $(x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)$ .

Hence the plane is the span of vectors  $\mathbf{v}_1 = (0, 1, 0)$  and  $\mathbf{v}_2 = (-2, 0, 1)$ . These vectors are linearly independent as they are not parallel.

Thus  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a basis for the plane  $x + 2z = 0$ .

**Theorem 1** Any vector space has a basis.

**Theorem 2** If a vector space  $V$  has a finite basis, then all bases for  $V$  are finite and have the same number of elements.

*Definition.* The **dimension** of a vector space  $V$ , denoted  $\dim V$ , is the number of elements in any of its bases.

*Examples.* •  $\dim \mathbb{R}^n = n$

•  $\mathcal{M}_{2,2}(\mathbb{R})$ : the space of  $2 \times 2$  matrices  
 $\dim \mathcal{M}_{2,2}(\mathbb{R}) = 4$

•  $\mathcal{M}_{m,n}(\mathbb{R})$ : the space of  $m \times n$  matrices  
 $\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

•  $\mathcal{P}_n$ : polynomials of degree less than  $n$   
 $\dim \mathcal{P}_n = n$

•  $\mathcal{P}$ : the space of all polynomials  
 $\dim \mathcal{P} = \infty$

•  $\{\mathbf{0}\}$ : the trivial vector space  
 $\dim \{\mathbf{0}\} = 0$

## How to find a basis?

**Theorem** Let  $S$  be a subset of a vector space  $V$ . Then the following conditions are equivalent:

- (i)  $S$  is a linearly independent spanning set for  $V$ , i.e., a basis;
- (ii)  $S$  is a minimal spanning set for  $V$ ;
- (iii)  $S$  is a maximal linearly independent subset of  $V$ .

“Minimal spanning set” means “remove any element from this set, and it is no longer a spanning set”.

“Maximal linearly independent subset” means “add any element of  $V$  to this set, and it will become linearly dependent”.

**Theorem** Let  $V$  be a vector space. Then

- (i) any spanning set for  $V$  can be reduced to a minimal spanning set;
- (ii) any linearly independent subset of  $V$  can be extended to a maximal linearly independent set.

Equivalently, any spanning set contains a basis, while any linearly independent set is contained in a basis.

**Corollary** A vector space is finite-dimensional if and only if it is spanned by a finite set.

## How to find a basis?

*Approach 1.* Get a spanning set for the vector space, then reduce this set to a basis.

**Proposition** Let  $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$  be a spanning set for a vector space  $V$ . If  $\mathbf{v}_0$  is a linear combination of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  then  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is also a spanning set for  $V$ .

Indeed, if  $\mathbf{v}_0 = r_1\mathbf{v}_1 + \dots + r_k\mathbf{v}_k$ , then

$$\begin{aligned} t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k &= \\ &= (t_0r_1 + t_1)\mathbf{v}_1 + \dots + (t_0r_k + t_k)\mathbf{v}_k. \end{aligned}$$



## How to find a basis?

*Approach 2.* Build a maximal linearly independent set adding one vector at a time.

If the vector space  $V$  is trivial, it has the empty basis.

If  $V \neq \{\mathbf{0}\}$ , pick any vector  $\mathbf{v}_1 \neq \mathbf{0}$ .

If  $\mathbf{v}_1$  spans  $V$ , it is a basis. Otherwise pick any vector  $\mathbf{v}_2 \in V$  that is not in the span of  $\mathbf{v}_1$ .

If  $\mathbf{v}_1$  and  $\mathbf{v}_2$  span  $V$ , they constitute a basis.

Otherwise pick any vector  $\mathbf{v}_3 \in V$  that is not in the span of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

And so on...

**Problem.** Find a basis for the vector space  $V$  spanned by vectors  $\mathbf{w}_1 = (1, 1, 0)$ ,  $\mathbf{w}_2 = (0, 1, 1)$ ,  $\mathbf{w}_3 = (2, 3, 1)$ , and  $\mathbf{w}_4 = (1, 1, 1)$ .

To pare this spanning set, we need to find a relation of the form  $r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + r_3\mathbf{w}_3 + r_4\mathbf{w}_4 = \mathbf{0}$ , where  $r_i \in \mathbb{R}$  are not all equal to zero. Equivalently,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this system of linear equations for  $r_1, r_2, r_3, r_4$ , we apply row reduction.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 \rightarrow \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{reduced row echelon form})$$

$$\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \iff \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}$$

General solution:  $(r_1, r_2, r_3, r_4) = (-2t, -t, t, 0)$ ,  $t \in \mathbb{R}$ .

Particular solution:  $(r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$ .

**Problem.** Find a basis for the vector space  $V$  spanned by vectors  $\mathbf{w}_1 = (1, 1, 0)$ ,  $\mathbf{w}_2 = (0, 1, 1)$ ,  $\mathbf{w}_3 = (2, 3, 1)$ , and  $\mathbf{w}_4 = (1, 1, 1)$ .

We have obtained that  $2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3 = \mathbf{0}$ .

Hence any of vectors  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$  can be dropped.

For instance,  $V = \text{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4)$ .

Let us check whether vectors  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4$  are linearly independent:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

They are!!! It follows that  $V = \mathbb{R}^3$  and  $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$  is a basis for  $V$ .

Vectors  $\mathbf{v}_1 = (0, 1, 0)$  and  $\mathbf{v}_2 = (-2, 0, 1)$  are linearly independent.

**Problem.** Extend the set  $\{\mathbf{v}_1, \mathbf{v}_2\}$  to a basis for  $\mathbb{R}^3$ .

Our task is to find a vector  $\mathbf{v}_3$  that is not a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

Then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  will be a basis for  $\mathbb{R}^3$ .

*Hint 1.*  $\mathbf{v}_1$  and  $\mathbf{v}_2$  span the plane  $x + 2z = 0$ .

The vector  $\mathbf{v}_3 = (1, 1, 1)$  does not lie in the plane  $x + 2z = 0$ , hence it is not a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . Thus  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis for  $\mathbb{R}^3$ .

Vectors  $\mathbf{v}_1 = (0, 1, 0)$  and  $\mathbf{v}_2 = (-2, 0, 1)$  are linearly independent.

**Problem.** Extend the set  $\{\mathbf{v}_1, \mathbf{v}_2\}$  to a basis for  $\mathbb{R}^3$ .

Our task is to find a vector  $\mathbf{v}_3$  that is not a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . Then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  will be a basis for  $\mathbb{R}^3$ .

*Hint 2.* At least one of vectors  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$ , and  $\mathbf{e}_3 = (0, 0, 1)$  is a desired one.

Let us check that  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$  and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$  are two bases for  $\mathbb{R}^3$ :

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \quad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$