MATH 304

Linear Algebra

Lecture 15:
Linear transformations (continued).
Range and kernel.
Matrix transformations.

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Properties of linear mappings

Let $L : V_1 \to V_2$ be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1 L(\mathbf{v}_1) + \cdots + r_k L(\mathbf{v}_k)$

for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.

$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1 L(\mathbf{v}_1) + r_2 L(\mathbf{v}_2)$,
$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) =$
$= r_1 L(\mathbf{v}_1) + r_2 L(\mathbf{v}_2) + r_3 L(\mathbf{v}_3)$, and so on.

- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.

$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2$.

- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v})$.

Examples of linear mappings

- *Scaling* $L : V \to V,\ L(\mathbf{v}) = s\mathbf{v},$ where $s \in \mathbb{R}$.

$L(\mathbf{x}+\mathbf{y}) = s(\mathbf{x}+\mathbf{y}) = s\mathbf{x}+s\mathbf{y} = L(\mathbf{x})+L(\mathbf{y}),$
$L(r\mathbf{x}) = s(r\mathbf{x}) = r(s\mathbf{x}) = rL(\mathbf{x}).$

- *Dot product with a fixed vector*

$\ell : \mathbb{R}^n \to \mathbb{R},\ \ell(\mathbf{v}) = \mathbf{v}\cdot\mathbf{v}_0,$ where $\mathbf{v}_0 \in \mathbb{R}^n$.

$\ell(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y})\cdot\mathbf{v}_0 = \mathbf{x}\cdot\mathbf{v}_0+\mathbf{y}\cdot\mathbf{v}_0 = \ell(\mathbf{x})+\ell(\mathbf{y}),$
$\ell(r\mathbf{x}) = (r\mathbf{x})\cdot\mathbf{v}_0 = r(\mathbf{x}\cdot\mathbf{v}_0) = r\ell(\mathbf{x}).$

- *Cross product with a fixed vector*

$L : \mathbb{R}^3 \to \mathbb{R}^3,\ L(\mathbf{v}) = \mathbf{v}\times\mathbf{v}_0,$ where $\mathbf{v}_0 \in \mathbb{R}^3$.

- *Multiplication by a fixed matrix*

$L : \mathbb{R}^n \to \mathbb{R}^m,\ L(\mathbf{v}) = A\mathbf{v},$ where A is an $m\times n$ matrix and all vectors are column vectors.

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$\ell : F(\mathbb{R}) \to \mathbb{R},\ \ \ell(f) = f(a),$ where $a \in \mathbb{R}$.

- *Multiplication by a fixed function*

$L : F(\mathbb{R}) \to F(\mathbb{R}),\ \ L(f) = gf,$ where $g \in F(\mathbb{R})$.

- *Differentiation* $D : C^1(\mathbb{R}) \to C(\mathbb{R}),\ \ L(f) = f'$.

$D(f+g) = (f+g)' = f' + g' = D(f) + D(g),$
$D(rf) = (rf)' = rf' = rD(f).$

- *Integration over a finite interval*

$\ell : C(\mathbb{R}) \to \mathbb{R},\ \ \ell(f) = \int_a^b f(x)\,dx,$ where
$a, b \in \mathbb{R},\ \ a < b.$

Properties of linear mappings

- If a linear mapping $L : V \longrightarrow W$ is invertible then the inverse mapping $L^{-1} : W \longrightarrow V$ is also linear.

- If $L : V \longrightarrow W$ and $M : W \longrightarrow X$ are linear mappings then the composition $M \circ L : V \longrightarrow X$ is also linear.

- If $L_1 : V \longrightarrow W$ and $L_2 : V \longrightarrow W$ are linear mappings then the sum $L_1 + L_2$ is also linear.

Linear differential operators

- an ordinary differential operator

$$L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on $\mathbb{R}$.
That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- Laplace's operator $\Delta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \;\mapsto\; \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

is a one-to-one correspondence between V and $\mathbb{R}^n$. This correspondence respects linear operations in V and in $\mathbb{R}^n$, i.e., it is a **linear transformation**.

Change of coordinates

Let V be a vector space of dimension n.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccccc} & & V & & \\ & {\scriptstyle g_1} \swarrow & & \searrow {\scriptstyle g_2} & \\ \mathbb{R}^n & & \longrightarrow & & \mathbb{R}^n \end{array}$$

The composition $g_2 \circ g_1^{-1}$ is a **linear** transformation of $\mathbb{R}^n$.
It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Example. Let V be the subspace of $C(\mathbb{R})$ spanned by the function $f(x) = xe^x + 1$ and its derivatives. Then $\dim V = 3$.

One basis for V is $v_1 = f$, $v_2 = f'$, $v_3 = f''$.
Another basis is $u_1(x) = xe^x$, $u_2(x) = e^x$, $u_3(x) = 1$.

$$v_1(x) = xe^x + 1 = u_1(x) + u_3(x),$$
$$v_2(x) = xe^x + e^x = u_1(x) + u_2(x),$$
$$v_3(x) = xe^x + 2e^x = u_1(x) + 2u_2(x).$$

Transition matrix from v_1, v_2, v_3 to u_1, u_2, u_3: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$.

Notice that

$$(u_1, u_2, u_3) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} = (v_1, v_2, v_3).$$

Range and kernel

Let V, W be vector spaces and $L : V \to W$ be a linear mapping.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L, denoted $\ker(L)$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem **(i)** The range of L is a subspace of W.
(ii) The kernel of L is a subspace of V.

Example. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The kernel $\ker(L)$ is the nullspace of the matrix.

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The range $L(\mathbb{R}^3)$ is the column space of the matrix.

Example. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The range of L is spanned by vectors $(1, 1, 1)$, $(0, 2, 0)$, and $(-1, -1, -1)$. It follows that $L(\mathbb{R}^3)$ is the plane spanned by $(1, 1, 1)$ and $(0, 1, 0)$.

To find $\ker(L)$, we apply row reduction to the matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $(x, y, z) \in \ker(L)$ if $x - z = y = 0$.
It follows that $\ker(L)$ is the line spanned by $(1, 0, 1)$.

Example. $L : C^3(\mathbb{R}) \to C(\mathbb{R}),\ L(u) = u''' - 2u'' + u'$.

According to the theory of differential equations, the initial value problem

$$\begin{cases} u'''(x) - 2u''(x) + u'(x) = g(x), \quad x \in \mathbb{R}, \\ u(a) = b_0, \\ u'(a) = b_1, \\ u''(a) = b_2 \end{cases}$$

has a unique solution for any $g \in C(\mathbb{R})$ and any $b_0, b_1, b_2 \in \mathbb{R}$. It follows that $L(C^3(\mathbb{R})) = C(\mathbb{R})$.

Also, the initial data evaluation $I(u) = (u(a), u'(a), u''(a))$, which is a linear mapping $I : C^3(\mathbb{R}) \to \mathbb{R}^3$, is one-to-one when restricted to $\ker(L)$. Hence $\dim \ker(L) = 3$.

It is easy to check that $L(xe^x) = L(e^x) = L(1) = 0$.
It follows that $\ker(L) = \mathrm{Span}(xe^x, e^x, 1)$.

General linear equations

Definition. A **linear equation** is an equation of the form

$$\boxed{L(\mathbf{x}) = \mathbf{b},}$$

where $L : V \to W$ is a linear mapping, $\mathbf{b}$ is a given vector from W, and $\mathbf{x}$ is an unknown vector from V.

The range of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The kernel of L is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable and $\dim \ker L < \infty$, then the general solution is

$$\mathbf{x}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k,$$

where $\mathbf{x}_0$ is a particular solution, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a basis for the kernel of L, and $t_1, \ldots, t_k$ are arbitrary scalars.

Example. $\begin{cases} x+y+z=4, \\ x+2y=3. \end{cases}$

$$L:\mathbb{R}^3\to\mathbb{R}^2,\quad L\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&1&1\\1&2&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

Linear equation: $L(\mathbf{x})=\mathbf{b}$, where $\mathbf{b}=\begin{pmatrix}4\\3\end{pmatrix}$.

$$\left(\begin{array}{ccc|c}1&1&1&4\\1&2&0&3\end{array}\right)\to\left(\begin{array}{ccc|c}\color{red}{1}&1&1&4\\0&\color{red}{1}&-1&-1\end{array}\right)\to\left(\begin{array}{ccc|c}\color{red}{1}&0&2&5\\0&\color{red}{1}&-1&-1\end{array}\right)$$

$$\begin{cases} x+2z=5 \\ y-z=-1 \end{cases}\iff\begin{cases} x=5-2z \\ y=-1+z \end{cases}$$

$$(x,y,z)=(5-2t,-1+t,t)=(5,-1,0)+t(-2,1,1).$$

Example. $u'''(x) - 2u''(x) + u'(x) = e^{2x}$.

Linear operator $L : C^3(\mathbb{R}) \to C(\mathbb{R})$,
$Lu = u''' - 2u'' + u'$.

Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

We already know that functions xe^x, e^x and 1 form a basis for the kernel of L. It remains to find a particular solution.

$L(e^{2x}) = 8e^{2x} - 2(4e^{2x}) + 2e^{2x} = 2e^{2x}$.
Since L is a linear operator, $L\left(\frac{1}{2}e^{2x}\right) = e^{2x}$.

Particular solution: $u_0(x) = \frac{1}{2}e^{2x}$.

Thus the general solution is

$$u(x) = \frac{1}{2}e^{2x} + t_1 xe^x + t_2 e^x + t_3.$$

Matrix transformations

Any $m \times n$ matrix A gives rise to a transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $L(\mathbf{x}) \in \mathbb{R}^m$ are regarded as column vectors. This transformation is **linear**.

Example. $$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis for $\mathbb{R}^3$. We have that $L(\mathbf{e}_1) = (1, 3, 0)$, $L(\mathbf{e}_2) = (0, 4, 5)$, $L(\mathbf{e}_3) = (2, 7, 8)$. Thus $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$ are columns of the matrix.

Problem. Find a linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(\mathbf{e}_1) = (1, 1)$, $L(\mathbf{e}_2) = (0, -2)$, $L(\mathbf{e}_3) = (3, 0)$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for $\mathbb{R}^3$.

$$L(x, y, z) = L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3)$$
$$= xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3)$$
$$= x(1, 1) + y(0, -2) + z(3, 0) = (x + 3z, x - 2y)$$

$$L(x, y, z) = \begin{pmatrix} x + 3z \\ x - 2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Columns of the matrix are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$.

Theorem Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for $\mathbb{R}^n$.

$$\mathbf{y} = A\mathbf{x} \quad \Longleftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$