MATH 304

Linear Algebra

Lecture 23: Diagonalization.

Review for Test 2.

Diagonalization

Let L be a linear operator on a finite-dimensional vector space V. Then the following conditions are equivalent:

- the matrix of L with respect to some basis is diagonal;
- there exists a basis for V formed by eigenvectors of L.

The operator L is **diagonalizable** if it satisfies these conditions.

Let A be an $n \times n$ matrix. Then the following conditions are equivalent:

- A is the matrix of a diagonalizable operator;
- A is similar to a diagonal matrix, i.e., it is represented as $A = UBU^{-1}$, where the matrix B is diagonal;
- there exists a basis for $\mathbb{R}^n$ formed by eigenvectors of A.

The matrix A is **diagonalizable** if it satisfies these conditions. Otherwise A is called **defective**.

Theorem 1 If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of a linear operator L associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Theorem 2 Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of a linear operator L. For any $1 \leq i \leq k$ let S_i be a basis for the eigenspace associated with the eigenvalue λ_i. Then the union $S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent set.

Corollary Let A be an $n \times n$ matrix such that the characteristic equation $\det(A - \lambda I) = 0$ has n distinct real roots. Then

(i) there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of A;

(ii) all eigenspaces of A are one-dimensional.

Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 1 and 3.
- The eigenspace of A associated with the eigenvalue 1 is the line spanned by $\mathbf{v}_1 = (-1, 1)$.
- The eigenspace of A associated with the eigenvalue 3 is the line spanned by $\mathbf{v}_2 = (1, 1)$.
- Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$.

Thus the matrix A is diagonalizable. Namely, $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Example. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 0 and 2.
- The eigenspace corresponding to 0 is spanned by $\mathbf{v}_1 = (-1, 1, 0)$.
- The eigenspace corresponding to 2 is spanned by $\mathbf{v}_2 = (1, 1, 0)$ and $\mathbf{v}_3 = (-1, 0, 1)$.
- Eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

Thus the matrix A is diagonalizable. Namely, $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem. Diagonalize the matrix $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$.

We need to find a diagonal matrix B and an invertible matrix U such that $A = UBU^{-1}$.
Suppose that $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ is a basis for $\mathbb{R}^2$ formed by eigenvectors of A, i.e., $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for some $\lambda_i \in \mathbb{R}$. Then we can take

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad U = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

Note that U is the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2$ to the standard basis.

Problem. Diagonalize the matrix $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$.

Characteristic equation of A: $\begin{vmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = 0.$

$(4-\lambda)(1-\lambda) = 0 \implies \lambda_1 = 4,\ \lambda_2 = 1.$

Associated eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Thus $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Problem. Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find A^5.

We know that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then $A^5 = UBU^{-1}UBU^{-1}UBU^{-1}UBU^{-1}UBU^{-1}$

$$= UB^5U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1024 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1024 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1024 & 1023 \\ 0 & 1 \end{pmatrix}.$$

Problem. Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find a matrix C such that $C^2 = A$.

We know that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Suppose that $D^2 = B$ for some matrix D. Let $C = UDU^{-1}$.
Then $C^2 = UDU^{-1}UDU^{-1} = UD^2U^{-1} = UBU^{-1} = A$.

We can take $D = \begin{pmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

Initial value problem for a system of linear ODEs:

$$\begin{cases} \frac{dx}{dt} = 4x + 3y, \\ \frac{dy}{dt} = y, \end{cases} \qquad x(0) = 1, \;\; y(0) = 1.$$

The system can be rewritten in vector form:

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v}, \quad \text{where} \;\; A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}, \;\; \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Matrix A is diagonalizable: $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be coordinates of the vector $\mathbf{v}$ relative to the basis $\mathbf{v}_1 = (1,0)^T$, $\mathbf{v}_2 = (-1,1)^T$ of eigenvectors of A. Then $\mathbf{v} = U\mathbf{w} \implies \mathbf{w} = U^{-1}\mathbf{v}$.

It follows that

$$\frac{d\mathbf{w}}{dt} = \frac{d}{dt}(U^{-1}\mathbf{v}) = U^{-1}\frac{d\mathbf{v}}{dt} = U^{-1}A\mathbf{v} = U^{-1}AU\mathbf{w}.$$

Hence $\quad \dfrac{d\mathbf{w}}{dt} = B\mathbf{w} \quad \Longleftrightarrow \quad \begin{cases} \frac{dw_1}{dt} = 4w_1, \\ \frac{dw_2}{dt} = w_2. \end{cases}$

General solution: $w_1(t) = c_1e^{4t}$, $w_2(t) = c_2e^{t}$, where $c_1, c_2 \in \mathbb{R}$.

Initial condition:

$$\mathbf{w}(0) = U^{-1}\mathbf{v}(0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus $w_1(t) = 2e^{4t}$, $w_2(t) = e^{t}$. Then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = U\mathbf{w}(t) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2e^{4t} \\ e^{t} \end{pmatrix} = \begin{pmatrix} 2e^{4t}-e^{t} \\ e^{t} \end{pmatrix}.$$

There are **two obstructions** to diagonalization. They are illustrated by the following examples.

Example 1. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\det(A - \lambda I) = (\lambda - 1)^2$. Hence $\lambda = 1$ is the only eigenvalue. The associated eigenspace is the line $t(1, 0)$.

Example 2. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = \lambda^2 + 1$.

$\implies$ no real eigenvalues or eigenvectors

(However there are *complex* eigenvalues/eigenvectors.)

Topics for Test 2

Coordinates and linear transformations (Leon 3.5, 4.1–4.3)

- Coordinates relative to a basis
- Change of basis, transition matrix
- Matrix transformations
- Matrix of a linear mapping

Orthogonality (Leon 5.1–5.6)

- Inner products and norms
- Orthogonal complement, orthogonal projection
- Least squares problems
- The Gram-Schmidt orthogonalization process

Eigenvalues and eigenvectors (Leon 6.1, 6.3)

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Diagonalization

Sample problems for Test 2

Problem 1 (15 pts.) Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the vector space of 2×2 matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Find the matrix of the operator L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 2 (20 pts.) Find a linear polynomial which is the best least squares fit to the following data:

x	-2	-1	0	1	2
$f(x)$	-3	-2	1	2	5

Problem 3 (25 pts.) Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

(i) Find an orthonormal basis for V.

(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.

Problem 4 (30 pts.) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.
(ii) For each eigenvalue of A, find an associated eigenvector.
(iii) Is the matrix A diagonalizable? Explain.
(iv) Find all eigenvalues of the matrix A^2.

Bonus Problem 5 (15 pts.) Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space V to a vector space W. Show that

$$\dim \mathrm{Range}(L) + \dim \ker(L) = \dim V.$$

Problem 1. Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the vector space of 2×2 matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Find the matrix of the operator L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let M_L denote the desired matrix.

By definition, M_L is a 4×4 matrix whose columns are coordinates of the matrices $L(E_1), L(E_2), L(E_3), L(E_4)$ with respect to the basis E_1, E_2, E_3, E_4.

$$L(E_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1E_1 + 2E_2 + 0E_3 + 0E_4,$$

$$L(E_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} = 3E_1 + 4E_2 + 0E_3 + 0E_4,$$

$$L(E_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = 0E_1 + 0E_2 + 1E_3 + 2E_4,$$

$$L(E_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = 0E_1 + 0E_2 + 3E_3 + 4E_4.$$

It follows that

$$M_L = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{pmatrix}.$$

Thus the relation

$$\begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

is equivalent to the relation

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Problem 2. Find a linear polynomial which is the best least squares fit to the following data:

x	-2	-1	0	1	2
$f(x)$	-3	-2	1	2	5

We are looking for a function $f(x) = c_1 + c_2 x$, where c_1, c_2 are unknown coefficients. The data of the problem give rise to an overdetermined system of linear equations in variables c_1 and c_2:

$$\begin{cases} c_1 - 2c_2 = -3, \\ c_1 - c_2 = -2, \\ c_1 = 1, \\ c_1 + c_2 = 2, \\ c_1 + 2c_2 = 5. \end{cases}$$

This system is inconsistent.

We can represent the system as a matrix equation $A\mathbf{c} = \mathbf{y}$, where

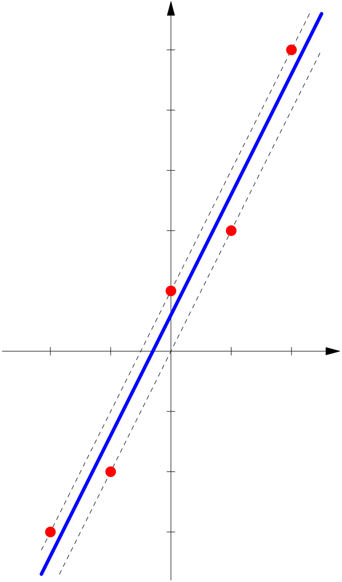
$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

The least squares solution $\mathbf{c}$ of the above system is a solution of the normal system $A^T A\mathbf{c} = A^T\mathbf{y}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\iff \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 20 \end{pmatrix} \iff \begin{cases} c_1 = 3/5 \\ c_2 = 2 \end{cases}$$

Thus the function $f(x) = \frac{3}{5} + 2x$ is the best least squares fit to the above data among linear polynomials.



Problem 3. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

(i) Find an orthonormal basis for V.

First we apply the Gram-Schmidt orthogonalization process to vectors $\mathbf{x}_1, \mathbf{x}_2$ and obtain an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2$ for the subspace V:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1, 1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1, 0, 3, 0) - \frac{4}{4}(1, 1, 1, 1) = (0, -1, 2, -1).$$

Then we normalize vectors $\mathbf{v}_1, \mathbf{v}_2$ to obtain an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2$ for V:

$$\|\mathbf{v}_1\| = 2 \implies \mathbf{w}_1 = \tfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \tfrac{1}{2}(1, 1, 1, 1)$$

$$\|\mathbf{v}_2\| = \sqrt{6} \implies \mathbf{w}_2 = \tfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \tfrac{1}{\sqrt{6}}(0, -1, 2, -1)$$

Problem 3. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.

(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.

Since the subspace V is spanned by vectors $(1,1,1,1)$ and $(1,0,3,0)$, it is the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

Then the orthogonal complement $V^\perp$ is the nullspace of A. To find the nullspace, we convert the matrix A to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

Hence a vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ belongs to $V^\perp$ if and only if

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \quad \begin{cases} x_1 + 3x_3 = 0 \\ x_2 - 2x_3 + x_4 = 0 \end{cases} \quad \iff \quad \begin{cases} x_1 = -3x_3 \\ x_2 = 2x_3 - x_4 \end{cases}$$

The general solution of the system is $(x_1, x_2, x_3, x_4) =$ $= (-3t, 2t - s, t, s) = t(-3, 2, 1, 0) + s(0, -1, 0, 1)$, where $t, s \in \mathbb{R}$.

It follows that $V^\perp$ is spanned by vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$.

The vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$ form a basis for the subspace $V^\perp$.

It remains to orthogonalize and normalize this basis:

$$\mathbf{v}_3 = \mathbf{x}_3 = (0, -1, 0, 1),$$

$$\mathbf{v}_4 = \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\,\mathbf{v}_3 = (-3, 2, 1, 0) - \frac{-2}{2}(0, -1, 0, 1)$$

$$= (-3, 1, 1, 1),$$

$$\|\mathbf{v}_3\| = \sqrt{2} \implies \mathbf{w}_3 = \tfrac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \tfrac{1}{\sqrt{2}}(0, -1, 0, 1),$$

$$\|\mathbf{v}_4\| = \sqrt{12} = 2\sqrt{3} \implies \mathbf{w}_4 = \tfrac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \tfrac{1}{2\sqrt{3}}(-3, 1, 1, 1).$$

Thus the vectors $\mathbf{w}_3 = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$ and $\mathbf{w}_4 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$ form an orthonormal basis for $V^\perp$.

Problem 3. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.
(i) Find an orthonormal basis for V.
(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.

Alternative solution: First we extend the set $\mathbf{x}_1, \mathbf{x}_2$ to a basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ for $\mathbb{R}^4$. Then we orthogonalize and normalize the latter. This yields an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ for $\mathbb{R}^4$.

By construction, $\mathbf{w}_1, \mathbf{w}_2$ is an orthonormal basis for V.
It follows that $\mathbf{w}_3, \mathbf{w}_4$ is an orthonormal basis for $V^\perp$.

The set $\mathbf{x}_1 = (1,1,1,1)$, $\mathbf{x}_2 = (1,0,3,0)$ can be extended to a basis for $\mathbb{R}^4$ by adding two vectors from the standard basis.

For example, we can add vectors $\mathbf{e}_3 = (0,0,1,0)$ and $\mathbf{e}_4 = (0,0,0,1)$. To show that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{e}_3, \mathbf{e}_4$ is indeed a basis for $\mathbb{R}^4$, we check that the matrix whose rows are these vectors is nonsingular:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0.$$

To orthogonalize the basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{e}_3, \mathbf{e}_4$, we apply the Gram-Schmidt process:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1, 1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1, 0, 3, 0) - \tfrac{4}{4}(1, 1, 1, 1) = (0, -1, 2, -1),$$

$$\mathbf{v}_3 = \mathbf{e}_3 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (0, 0, 1, 0) - \tfrac{1}{4}(1, 1, 1, 1) -$$

$$-\tfrac{2}{6}(0, -1, 2, -1) = \left(-\tfrac{1}{4}, \tfrac{1}{12}, \tfrac{1}{12}, \tfrac{1}{12}\right) = \tfrac{1}{12}(-3, 1, 1, 1),$$

$$\mathbf{v}_4 = \mathbf{e}_4 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 = (0, 0, 0, 1) -$$

$$-\tfrac{1}{4}(1, 1, 1, 1) - \tfrac{-1}{6}(0, -1, 2, -1) - \tfrac{1/12}{1/12} \cdot \tfrac{1}{12}(-3, 1, 1, 1) =$$

$$= \left(0, -\tfrac{1}{2}, 0, \tfrac{1}{2}\right) = \tfrac{1}{2}(0, -1, 0, 1).$$

It remains to normalize vectors $\mathbf{v}_1 = (1,1,1,1)$, $\mathbf{v}_2 = (0,-1,2,-1)$, $\mathbf{v}_3 = \frac{1}{12}(-3,1,1,1)$, $\mathbf{v}_4 = \frac{1}{2}(0,-1,0,1)$:

$$\|\mathbf{v}_1\| = 2 \implies \mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}(1,1,1,1)$$

$$\|\mathbf{v}_2\| = \sqrt{6} \implies \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}(0,-1,2,-1)$$

$$\|\mathbf{v}_3\| = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} \implies \mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{2\sqrt{3}}(-3,1,1,1)$$

$$\|\mathbf{v}_4\| = \frac{1}{\sqrt{2}} \implies \mathbf{w}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{1}{\sqrt{2}}(0,-1,0,1)$$

Thus $\mathbf{w}_1, \mathbf{w}_2$ is an orthonormal basis for V while $\mathbf{w}_3, \mathbf{w}_4$ is an orthonormal basis for $V^\perp$.

Problem 4. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.

The eigenvalues of A are roots of the characteristic equation $\det(A - \lambda I) = 0$. We obtain that

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^3 - 2(1-\lambda) - 2(1-\lambda) = (1-\lambda)\big((1-\lambda)^2 - 4\big)$$

$$= (1-\lambda)\big((1-\lambda) - 2\big)\big((1-\lambda) + 2\big) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix A has three eigenvalues: -1, 1, and 3.

Problem 4. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(ii) For each eigenvalue of A, find an associated eigenvector.

An eigenvector $\mathbf{v} = (x, y, z)$ of the matrix A associated with an eigenvalue λ is a nonzero solution of the vector equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve the equation, we convert the matrix $A - \lambda I$ to reduced row echelon form.

First consider the case $\lambda = -1$. The row reduction yields

$$A + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A + I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)$ is an eigenvector of A associated with the eigenvalue -1.

Secondly, consider the case $\lambda = 1$. The row reduction yields

$$A - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - I)\mathbf{v} = \mathbf{0} \quad \iff \quad \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, 0, 1)$ is an eigenvector of A associated with the eigenvalue 1.

Finally, consider the case $\lambda = 3$. The row reduction yields

$$A-3I = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A-3I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = t,\ y = t,\ z = t,$ where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1,1,1)$ is an eigenvector of A associated with the eigenvalue 3.

Problem 4. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(iii) Is the matrix A diagonalizable? Explain.

The matrix A is diagonalizable, i.e., there exists a basis for $\mathbb{R}^3$ formed by its eigenvectors.

Namely, the vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix A belonging to distinct eigenvalues. Therefore these vectors are linearly independent. It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

Alternatively, the existence of a basis for $\mathbb{R}^3$ consisting of eigenvectors of A already follows from the fact that the matrix A has three distinct eigenvalues.

Problem 4. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(iv) Find all eigenvalues of the matrix A^2.

Suppose that $\mathbf{v}$ is an eigenvector of the matrix A associated with an eigenvalue λ, that is, $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

Therefore $\mathbf{v}$ is also an eigenvector of the matrix A^2 and the associated eigenvalue is λ^2. We already know that the matrix A has eigenvalues -1, 1, and 3. It follows that A^2 has eigenvalues 1 and 9.

Since a 3×3 matrix can have up to 3 eigenvalues, we need an additional argument to show that 1 and 9 are the only eigenvalues of A^2. One reason is that the eigenvalue 1 has multiplicity 2.

Bonus Problem 5. Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space V to a vector space W. Show that $\dim \mathrm{Range}(L) + \dim \ker(L) = \dim V$.

The kernel $\ker(L)$ is a subspace of V. It is finite-dimensional since the vector space V is.

Take a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ for the subspace $\ker(L)$, then extend it to a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ for the entire space V.

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range of L.

Assuming the claim is proved, we obtain

$$\dim \mathrm{Range}(L) = m, \quad \dim \ker(L) = k, \quad \dim V = k + m.$$

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range of L.

Proof (spanning): Any vector $\mathbf{w} \in \mathrm{Range}(L)$ is represented as $\mathbf{w} = L(\mathbf{v})$, where $\mathbf{v} \in V$. Then

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_m\mathbf{u}_m$$

for some $\alpha_i, \beta_j \in \mathbb{R}$. It follows that

$$\begin{aligned}\mathbf{w} = L(\mathbf{v}) &= \alpha_1 L(\mathbf{v}_1) + \cdots + \alpha_k L(\mathbf{v}_k) + \beta_1 L(\mathbf{u}_1) + \cdots + \beta_m L(\mathbf{u}_m) \\ &= \beta_1 L(\mathbf{u}_1) + \cdots + \beta_m L(\mathbf{u}_m).\end{aligned}$$

Note that $L(\mathbf{v}_i) = \mathbf{0}$ since $\mathbf{v}_i \in \ker(L)$.
Thus $\mathrm{Range}(L)$ is spanned by the vectors $L(\mathbf{u}_1), \dots, L(\mathbf{u}_m)$.

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range of L.

Proof (linear independence): Suppose that

$$t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \cdots + t_m L(\mathbf{u}_m) = \mathbf{0}$$

for some $t_i \in \mathbb{R}$. Let $\mathbf{u} = t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_m\mathbf{u}_m$. Since

$$L(\mathbf{u}) = t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \cdots + t_m L(\mathbf{u}_m) = \mathbf{0},$$

the vector $\mathbf{u}$ belongs to the kernel of L. Therefore $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k$ for some $s_j \in \mathbb{R}$. It follows that

$$t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_m\mathbf{u}_m - s_1\mathbf{v}_1 - s_2\mathbf{v}_2 - \cdots - s_k\mathbf{v}_k = \mathbf{u} - \mathbf{u} = \mathbf{0}.$$

Linear independence of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m$ implies that $t_1 = \cdots = t_m = 0$ (as well as $s_1 = \cdots = s_k = 0$).
Thus the vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ are linearly independent.