

## Sample problems for the final exam: Solutions

Any problem may be altered or replaced by a different one!

**Problem 1 (15 pts.)** Find a quadratic polynomial  $p(x)$  such that  $p(-1) = p(3) = 6$  and  $p'(2) = p(1)$ .

Let  $p(x) = ax^2 + bx + c$ . Then  $p(-1) = a - b + c$ ,  $p(1) = a + b + c$ , and  $p(3) = 9a + 3b + c$ . Also,  $p'(x) = 2ax + b$  and  $p'(2) = 4a + b$ . The coefficients  $a$ ,  $b$ , and  $c$  have to be chosen so that

$$\begin{cases} a - b + c = 6, \\ 9a + 3b + c = 6, \\ 4a + b = a + b + c. \end{cases}$$

This is a system of linear equations in variables  $a, b, c$ . To solve the system, let us convert the third equation to the standard form and add it to the first and the second equations:

$$\begin{cases} a - b + c = 6 \\ 9a + 3b + c = 6 \\ 3a - c = 0 \end{cases} \iff \begin{cases} 4a - b = 6 \\ 9a + 3b + c = 6 \\ 3a - c = 0 \end{cases} \iff \begin{cases} 4a - b = 6 \\ 12a + 3b = 6 \\ 3a - c = 0 \end{cases}$$

Now divide the second equation by 3, add it to the first equation, and find the solution by back substitution:

$$\begin{cases} 4a - b = 6 \\ 4a + b = 2 \\ 3a - c = 0 \end{cases} \iff \begin{cases} 8a = 8 \\ 4a + b = 2 \\ 3a - c = 0 \end{cases} \iff \begin{cases} a = 1 \\ 4a + b = 2 \\ 3a - c = 0 \end{cases} \iff \begin{cases} a = 1 \\ b = -2 \\ 3a - c = 0 \end{cases} \iff \begin{cases} a = 1 \\ b = -2 \\ c = 3 \end{cases}$$

Thus the desired polynomial is  $p(x) = x^2 - 2x + 3$ .

**Problem 2 (20 pts.)** Let  $\mathbf{v}_1 = (1, 1, 1)$ ,  $\mathbf{v}_2 = (1, 1, 0)$ , and  $\mathbf{v}_3 = (1, 0, 1)$ . Let  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear operator on  $\mathbb{R}^3$  such that  $L(\mathbf{v}_1) = \mathbf{v}_2$ ,  $L(\mathbf{v}_2) = \mathbf{v}_3$ ,  $L(\mathbf{v}_3) = \mathbf{v}_1$ .

(i) Show that the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  form a basis for  $\mathbb{R}^3$ .

Let  $U$  be a  $3 \times 3$  matrix such that its columns are vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ :

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To find the determinant of  $U$ , we subtract the second row from the first one and then expand by the first row:

$$\det U = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Since  $\det U \neq 0$ , the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent. It follows that they form a basis for  $\mathbb{R}^3$ .

(ii) Find the matrix of the operator  $L$  relative to the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ .

Let  $A$  denote the matrix of  $L$  relative to the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ . By definition, the columns of  $A$  are coordinates of vectors  $L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)$  with respect to the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ . Since  $L(\mathbf{v}_1) = \mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3$ ,  $L(\mathbf{v}_2) = \mathbf{v}_3 = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_3$ ,  $L(\mathbf{v}_3) = \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$ , we obtain

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(iii) Find the matrix of the operator  $L$  relative to the standard basis.

Let  $S$  denote the matrix of  $L$  relative to the standard basis for  $\mathbb{R}^3$ . We have  $S = UAU^{-1}$ , where  $A$  is the matrix of  $L$  relative to the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  (already found) and  $U$  is the transition matrix from  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  to the standard basis (the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are consecutive columns of  $U$ ):

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To find the inverse  $U^{-1}$ , we merge the matrix  $U$  with the identity matrix  $I$  into one  $3 \times 6$  matrix and apply row reduction to convert the left half  $U$  of this matrix into  $I$ . Simultaneously, the right half  $I$  will be converted into  $U^{-1}$ :

$$\begin{aligned} (U|I) &= \left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array} \right) \\ &\rightarrow \left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array} \right) \\ &\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array} \right) = (I|U^{-1}). \end{aligned}$$

Thus

$$\begin{aligned} S &= UAU^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}. \end{aligned}$$

*Alternative solution:* Let  $S$  denote the matrix of  $L$  relative to the standard basis  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$ ,  $\mathbf{e}_3 = (0, 0, 1)$ . By definition, the columns of  $S$  are vectors  $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$ . It is easy to observe that  $\mathbf{e}_2 = \mathbf{v}_1 - \mathbf{v}_3$ ,  $\mathbf{e}_3 = \mathbf{v}_1 - \mathbf{v}_2$ , and  $\mathbf{e}_1 = \mathbf{v}_2 - \mathbf{e}_2 = -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ . Therefore

$$\begin{aligned} L(\mathbf{e}_1) &= L(-\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = -L(\mathbf{v}_1) + L(\mathbf{v}_2) + L(\mathbf{v}_3) = -\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_1 = (1, 0, 2), \\ L(\mathbf{e}_2) &= L(\mathbf{v}_1 - \mathbf{v}_3) = L(\mathbf{v}_1) - L(\mathbf{v}_3) = \mathbf{v}_2 - \mathbf{v}_1 = (0, 0, -1), \\ L(\mathbf{e}_3) &= L(\mathbf{v}_1 - \mathbf{v}_2) = L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{v}_2 - \mathbf{v}_3 = (0, 1, -1). \end{aligned}$$

Thus

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

**Problem 3 (20 pts.)** Let  $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}$ .

(i) Evaluate the determinant of the matrix  $A$ .

The determinant of  $A$  is easily evaluated using column expansions:

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1.$$

Another way to evaluate  $\det A$  is to convert the matrix  $A$  into the identity matrix using elementary row operations (see below). This requires much more work but we are going to do it anyway, to find the inverse of  $A$ .

(ii) Find the inverse matrix  $A^{-1}$ .

First we merge the matrix  $A$  with the identity matrix into one  $4 \times 8$  matrix

$$(A|I) = \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract the first row from the second row:

$$\left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

Subtract 2 times the first row from the fourth row:

$$\left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \end{array} \right).$$

Interchange the second row with the fourth row:

$$\left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{array} \right).$$

Subtract the second row from the third row:

$$\left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{array} \right).$$

Interchange the third row with the fourth row:

$$\left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{array} \right).$$

Add the fourth row to the third row:

$$\left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{array} \right).$$

Subtract the second row from the first row:

$$\left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{array} \right).$$

Finally the left part of our  $4 \times 8$  matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of  $A$ . Thus

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 0 & 0 & -1 \\ -2 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 2 & 0 & 1 & -1 \end{pmatrix}.$$

As a byproduct, we can evaluate the determinant of  $A$ . We have transformed  $A$  into the identity matrix using elementary row operations. These included two row exchanges and no row multiplications. It follows that  $\det A = \det I = 1$ .

**Problem 4 (25 pts.)** Let  $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ .

(i) Find all eigenvalues of the matrix  $B$ .

The eigenvalues of  $B$  are roots of the characteristic equation  $\det(B - \lambda I) = 0$ . We obtain that

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} 1 - \lambda & 1 & 1 \\ 1 & 1 - \lambda & 1 \\ 1 & 1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^3 - 3(1 - \lambda) + 2 \\ &= (1 - 3\lambda + 3\lambda^2 - \lambda^3) - 3(1 - \lambda) + 2 = 3\lambda^2 - \lambda^3 = \lambda^2(3 - \lambda). \end{aligned}$$

Hence the matrix  $B$  has two eigenvalues: 0 and 3.

(ii) Find a basis for  $\mathbb{R}^3$  consisting of eigenvectors of  $B$ .

An eigenvector  $\mathbf{x} = (x, y, z)$  of  $B$  associated with an eigenvalue  $\lambda$  is a nonzero solution of the vector equation  $(B - \lambda I)\mathbf{x} = \mathbf{0}$ . First consider the case  $\lambda = 0$ . We obtain that

$$B\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x + y + z = 0.$$

The general solution is  $x = -t - s$ ,  $y = t$ ,  $z = s$ , where  $t, s \in \mathbb{R}$ . Equivalently,  $\mathbf{x} = t(-1, 1, 0) + s(-1, 0, 1)$ . Hence the eigenspace of  $B$  associated with the eigenvalue 0 is two-dimensional. It is spanned by eigenvectors  $\mathbf{v}_1 = (-1, 1, 0)$  and  $\mathbf{v}_2 = (-1, 0, 1)$ .

Now consider the case  $\lambda = 3$ . We obtain that

$$\begin{aligned} (B - 3I)\mathbf{x} = \mathbf{0} &\iff \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases} \end{aligned}$$

The general solution is  $x = y = z = t$ , where  $t \in \mathbb{R}$ . In particular,  $\mathbf{v}_3 = (1, 1, 1)$  is an eigenvector of  $B$  associated with the eigenvalue 3.

The vectors  $\mathbf{v}_1 = (-1, 1, 0)$ ,  $\mathbf{v}_2 = (-1, 0, 1)$ , and  $\mathbf{v}_3 = (1, 1, 1)$  are eigenvectors of the matrix  $B$ . They are linearly independent since the matrix whose rows are these vectors is nonsingular:

$$\begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3 \neq 0.$$

It follows that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  is a basis for  $\mathbb{R}^3$ .

(iii) Find an orthonormal basis for  $\mathbb{R}^3$  consisting of eigenvectors of  $B$ .

It is easy to check that the vector  $\mathbf{v}_3$  is orthogonal to  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . To transform the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  into an orthogonal one, we only need to orthogonalize the pair  $\mathbf{v}_1, \mathbf{v}_2$ . Using the Gram-Schmidt process, we replace the vector  $\mathbf{v}_2$  by

$$\mathbf{u} = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = (-1, 0, 1) - \frac{1}{2}(-1, 1, 0) = (-1/2, -1/2, 1).$$

Now  $\mathbf{v}_1, \mathbf{u}, \mathbf{v}_3$  is an orthogonal basis for  $\mathbb{R}^3$ . Since  $\mathbf{u}$  is a linear combination of the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , it is also an eigenvector of  $B$  associated with the eigenvalue 0.

Finally, vectors  $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$ ,  $\mathbf{w}_2 = \frac{\mathbf{u}}{\|\mathbf{u}\|}$ , and  $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$  form an orthonormal basis for  $\mathbb{R}^3$  consisting of eigenvectors of  $B$ . We get that  $\|\mathbf{v}_1\| = \sqrt{2}$ ,  $\|\mathbf{u}\| = \sqrt{3/2}$ , and  $\|\mathbf{v}_3\| = \sqrt{3}$ . Thus

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}(-1, 1, 0), \quad \mathbf{w}_2 = \frac{1}{\sqrt{6}}(-1, -1, 2), \quad \mathbf{w}_3 = \frac{1}{\sqrt{3}}(1, 1, 1).$$

(iv) Find a diagonal matrix  $X$  and an invertible matrix  $U$  such that  $B = UXU^{-1}$ .

The vectors  $\mathbf{v}_1 = (-1, 1, 0)$ ,  $\mathbf{v}_2 = (-1, 0, 1)$ , and  $\mathbf{v}_3 = (1, 1, 1)$  are eigenvectors of the matrix  $B$  associated with eigenvalues 0, 0, and 3, respectively. Since these vectors form a basis for  $\mathbb{R}^3$ , it follows that  $B = UXU^{-1}$ , where

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Here  $U$  is the transition matrix from the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  to the standard basis (its columns are vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ ) while  $X$  is the matrix of the linear operator  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $L(\mathbf{x}) = B\mathbf{x}$  with respect to the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ .

**Problem 5 (20 pts.)** Let  $V$  be a subspace of  $\mathbb{R}^4$  spanned by vectors  $\mathbf{x}_1 = (1, 1, 0, 0)$ ,  $\mathbf{x}_2 = (2, 0, -1, 1)$ , and  $\mathbf{x}_3 = (0, 1, 1, 0)$ .

- (i) Find the distance from the point  $\mathbf{y} = (0, 0, 0, 4)$  to the subspace  $V$ .  
(ii) Find the distance from the point  $\mathbf{y}$  to the orthogonal complement  $V^\perp$ .

The vector  $\mathbf{y}$  is uniquely represented as  $\mathbf{y} = \mathbf{p} + \mathbf{o}$ , where  $\mathbf{p} \in V$  and  $\mathbf{o}$  is orthogonal to  $V$ , that is,  $\mathbf{o} \in V^\perp$ . The vector  $\mathbf{p}$  is the orthogonal projection of  $\mathbf{y}$  onto the subspace  $V$ . Since  $(V^\perp)^\perp = V$ , the vector  $\mathbf{o}$  is the orthogonal projection of  $\mathbf{y}$  onto the subspace  $V^\perp$ . It follows that the distance from the point  $\mathbf{y}$  to  $V$  equals  $\|\mathbf{o}\|$  while the distance from  $\mathbf{y}$  to  $V^\perp$  equals  $\|\mathbf{p}\|$ .

The orthogonal projection  $\mathbf{p}$  of the vector  $\mathbf{y}$  onto the subspace  $V$  is easily computed when we have an orthogonal basis for  $V$ . To get such a basis, we apply the Gram-Schmidt orthogonalization process to the basis  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ :

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{x}_1 = (1, 1, 0, 0), & \mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = (2, 0, -1, 1) - \frac{2}{2}(1, 1, 0, 0) = (1, -1, -1, 1), \\ \mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 = (0, 1, 1, 0) - \frac{1}{2}(1, 1, 0, 0) - \frac{-2}{4}(1, -1, -1, 1) = (0, 0, 1/2, 1/2). \end{aligned}$$

Now that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  is an orthogonal basis for  $V$  we obtain

$$\begin{aligned} \mathbf{p} &= \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 + \frac{\mathbf{y} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3} \mathbf{v}_3 = \\ &= \frac{0}{2}(1, 1, 0, 0) + \frac{4}{4}(1, -1, -1, 1) + \frac{2}{1/2}(0, 0, 1/2, 1/2) = (1, -1, 1, 3). \end{aligned}$$

Consequently,  $\mathbf{o} = \mathbf{y} - \mathbf{p} = (0, 0, 0, 4) - (1, -1, 1, 3) = (-1, 1, -1, 1)$ . Thus the distance from  $\mathbf{y}$  to the subspace  $V$  equals  $\|\mathbf{o}\| = 2$  and the distance from  $\mathbf{y}$  to  $V^\perp$  equals  $\|\mathbf{p}\| = \sqrt{12} = 2\sqrt{3}$ .

**Bonus Problem 6 (15 pts.)** (i) Find a matrix exponential  $\exp(tC)$ , where  $C = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$  and  $t \in \mathbb{R}$ .

Observe that  $C = 3I + Y$ , where  $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $I$  is the identity matrix. Then  $tC = 3tI + tY$  for all  $t \in \mathbb{R}$ . Clearly, matrices  $3tI$  and  $tY$  commute:  $(3tI)(tY) = (tY)(3tI) = 3t^2Y$ . It follows that  $\exp(tC) = \exp(3tI)\exp(tY)$ . For any square matrix  $X$ ,

$$\exp(X) = I + X + \frac{1}{2!}X^2 + \cdots + \frac{1}{n!}X^n + \cdots$$

In particular,

$$\exp(3tI) = I + 3tI + \frac{1}{2!}(3tI)^2 + \cdots + \frac{1}{n!}(3tI)^n + \cdots = \left(1 + 3t + \frac{(3t)^2}{2!} + \cdots + \frac{(3t)^n}{n!} + \cdots\right) I = e^{3t}I.$$

Further notice that  $Y^2 = O$ . Then  $Y^n = O$  for any integer  $n \geq 2$ . Consequently,

$$\exp(tY) = I + tY = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Finally,

$$\exp(tC) = \exp(3tI) \exp(tY) = e^{3t}I \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{pmatrix}.$$

(ii) Solve a system of differential equations

$$\begin{cases} \frac{dx}{dt} = 3x + y, \\ \frac{dy}{dt} = 3y \end{cases}$$

subject to the initial conditions  $x(0) = y(0) = 1$ .

The system can be rewritten as  $\mathbf{v}' = C\mathbf{v}$ , where

$$\mathbf{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

The initial value problem has a unique solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tC} \mathbf{v}_0, \quad \text{where } \mathbf{v}_0 = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By the above

$$e^{tC} = \begin{pmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{pmatrix}.$$

Thus  $x(t) = e^{3t}(1 + t)$ ,  $y(t) = e^{3t}$ .

**Bonus Problem 7 (15 pts.)** Consider a linear operator  $K : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by

$$K(\mathbf{x}) = D\mathbf{x}, \quad \text{where } D = \frac{1}{9} \begin{pmatrix} -4 & 7 & 4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}.$$

The operator  $K$  is a rotation about an axis.

(i) Find the axis of rotation.

The axis of rotation is the set of points fixed by the operator  $K$ . Hence a point  $\mathbf{x} \in \mathbb{R}^3$  lies on the axis if and only if  $K(\mathbf{x}) = \mathbf{x}$  or, equivalently,  $(D - I)\mathbf{x} = \mathbf{0}$ . To solve this vector equation, we convert the matrix  $D - I$  to reduced row echelon form:

$$D - I = \frac{1}{9} \begin{pmatrix} -13 & 7 & 4 \\ 1 & -13 & 8 \\ 8 & 4 & -8 \end{pmatrix} \rightarrow \frac{1}{9} \begin{pmatrix} 3 & 15 & -12 \\ 1 & -13 & 8 \\ 8 & 4 & -8 \end{pmatrix} \rightarrow \frac{1}{9} \begin{pmatrix} 3 & 15 & -12 \\ 9 & -9 & 0 \\ 8 & 4 & -8 \end{pmatrix}$$

$$\begin{aligned} &\rightarrow \begin{pmatrix} 1 & 5 & -4 \\ 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 1 & 5 & -4 \\ 2 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 6 & -4 \\ 2 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 6 & -4 \\ 0 & 3 & -2 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Here is the list of performed operations: add 2 times the third row to the first row, add the third row to the second row, multiply the first row by 3 and the third row by 9/4, interchange the first row with the second row, subtract the first row from the second row, subtract 2 times the first row from the third row, subtract 2 times the third row from the second row, interchange the second row with the third row, multiply the second row by 1/3, and add the second row to the first row.

Therefore a point  $\mathbf{x} = (x, y, z)$  lies on the axis if and only if  $x - \frac{2}{3}z = y - \frac{2}{3}z = 0$ . The general solution of the system is  $x = y = \frac{2}{3}t$ ,  $z = t$ , where  $t \in \mathbb{R}$ . Thus the axis of rotation is the line spanned by the vector  $(2, 2, 3)$ .

(ii) Find the angle of rotation.

The trace (the sum of diagonal entries) of any square matrix is equal to the sum of its eigenvalues (including complex eigenvalues and counting with multiplicities). Since  $D$  is a rotation matrix, it is similar to the matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix},$$

where  $\phi$  is the angle of rotation. The matrix  $E$  has eigenvalues 1,  $e^{i\phi}$ , and  $e^{-i\phi}$ . Since the matrix  $D$  is similar to  $E$ , it has the same eigenvalues. It follows that the trace of  $D$ , which is  $-7/9$ , is equal to

$$1 + e^{i\phi} + e^{-i\phi} = 1 + (\cos \phi + i \sin \phi) + (\cos \phi - i \sin \phi) = 1 + 2 \cos \phi.$$

Therefore  $\cos \phi = -8/9$ . Thus  $\phi = \arccos(-8/9)$ .