MATH 304
Linear Algebra

Lecture 12:
Subspaces of vector spaces.
A vector space is a set $V$ equipped with two operations, **addition**

$$V \times V \ni (x, y) \mapsto x + y \in V$$

and **scalar multiplication**

$$\mathbb{R} \times V \ni (r, x) \mapsto rx \in V,$$

that have the following properties:
Properties of addition and scalar multiplication

A1. \( a + b = b + a \) for all \( a, b \in V \).

A2. \((a + b) + c = a + (b + c)\) for all \( a, b, c \in V \).

A3. There exists an element of \( V \), called the zero vector and denoted \( 0 \), such that \( a + 0 = 0 + a = a \) for all \( a \in V \).

A4. For any \( a \in V \) there exists an element of \( V \), denoted \(-a\), such that \( a + (-a) = (-a) + a = 0 \).

A5. \( r(a + b) = ra + rb \) for all \( r \in \mathbb{R} \) and \( a, b \in V \).

A6. \((r + s)a = ra + sa \) for all \( r, s \in \mathbb{R} \) and \( a \in V \).

A7. \((rs)a = r(sa) \) for all \( r, s \in \mathbb{R} \) and \( a \in V \).

A8. \( 1a = a \) for all \( a \in V \).
• Associativity of addition implies that a multiple sum \( u_1 + u_2 + \cdots + u_k \) is well defined for any \( u_1, u_2, \ldots, u_k \in V \).

• **Subtraction** in \( V \) is defined as usual: \( a - b = a + (-b) \).

• Addition and scalar multiplication are called **linear operations**.

Given \( u_1, u_2, \ldots, u_k \in V \) and \( r_1, r_2, \ldots, r_k \in \mathbb{R} \),

\[
    r_1 u_1 + r_2 u_2 + \cdots + r_k u_k
\]

is called a **linear combination** of \( u_1, u_2, \ldots, u_k \).
Additional properties of vector spaces

• The zero vector is unique.
• For any \( a \in V \), the negative \( -a \) is unique.

\[
\begin{align*}
\text{• } a + b = c & \iff a = c - b \quad \text{for all } a, b, c \in V. \\
\text{• } a + c = b + c & \iff a = b \quad \text{for all } a, b, c \in V. \\
\text{• } 0a = 0 & \quad \text{for any } a \in V. \\
\text{• } (-1)a = -a & \quad \text{for any } a \in V.
\end{align*}
\]
Examples of vector spaces

- $\mathbb{R}^n$: $n$-dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m \times n$ matrices with real entries
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \ldots)$, $x_i \in \mathbb{R}$
- $\{0\}$: the trivial vector space
- $F(\mathbb{R})$: the set of all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$
- $C^1(\mathbb{R})$: all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$
- $C^\infty(\mathbb{R})$: all smooth functions $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1x + \cdots + a_nx^n$
Subspaces of vector spaces

**Definition.** A vector space \( V_0 \) is a **subspace** of a vector space \( V \) if \( V_0 \subset V \) and the linear operations on \( V_0 \) agree with the linear operations on \( V \).

**Examples.**
- \( F(\mathbb{R}) \): all functions \( f : \mathbb{R} \to \mathbb{R} \)
- \( C(\mathbb{R}) \): all continuous functions \( f : \mathbb{R} \to \mathbb{R} \)
  
  \( C(\mathbb{R}) \) is a subspace of \( F(\mathbb{R}) \).
- \( \mathcal{P} \): polynomials \( p(x) = a_0 + a_1x + \cdots + a_kx^k \)
- \( \mathcal{P}_n \): polynomials of degree less than \( n \)
  
  \( \mathcal{P}_n \) is a subspace of \( \mathcal{P} \).
Subspaces of vector spaces

Counterexamples.

- $\mathbb{R}^n$: $n$-dimensional coordinate vectors
- $\mathbb{Q}^n$: vectors with rational coordinates

$\mathbb{Q}^n$ is not a subspace of $\mathbb{R}^n$.

$\sqrt{2}(1, 1, \ldots, 1) \notin \mathbb{Q}^n \implies \mathbb{Q}^n$ is not a vector space (scaling is not well defined).

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1x + \cdots + a_nx^n$
- $\mathcal{P}^*_n$: polynomials of degree $n$ ($n > 0$)

$\mathcal{P}^*_n$ is not a subspace of $\mathcal{P}$.

$-x^n + (x^n + 1) = 1 \notin \mathcal{P}^*_n \implies \mathcal{P}^*_n$ is not a vector space (addition is not well defined).
If $S$ is a subset of a vector space $V$ then $S$ inherits from $V$ addition and scalar multiplication. However $S$ need not be closed under these operations.

**Proposition** A subset $S$ of a vector space $V$ is a subspace of $V$ if and only if $S$ is **nonempty** and **closed under linear operations**, i.e.,

\[
\begin{align*}
\forall x, y \in S & \implies x + y \in S, \\
\forall x \in S & \implies rx \in S \text{ for all } r \in \mathbb{R}.
\end{align*}
\]

**Proof:** “only if” is obvious.
“if”: properties like associative, commutative, or distributive law hold for $S$ because they hold for $V$. We only need to verify properties A3 and A4. Take any $x \in S$ (note that $S$ is nonempty). Then $0 = 0x \in S$. Also, $-x = (-1)x \in S$. 
Example. \( V = \mathbb{R}^2 \).

- The line \( x - y = 0 \) is a subspace of \( \mathbb{R}^2 \).

The line consists of all vectors of the form \( (t, t) \), \( t \in \mathbb{R} \).

\[
(t, t) + (s, s) = (t + s, t + s) \implies \text{closed under addition}
\]

\[
r(t, t) = (rt, rt) \implies \text{closed under scaling}
\]

- The parabola \( y = x^2 \) is not a subspace of \( \mathbb{R}^2 \).

It is enough to find one explicit counterexample.

Counterexample 1: \( (1, 1) + (-1, 1) = (0, 2) \).

\( (1, 1) \) and \( (-1, 1) \) lie on the parabola while \( (0, 2) \) does not

\implies \text{not closed under addition}

Counterexample 2: \( 2(1, 1) = (2, 2) \).

\( (1, 1) \) lies on the parabola while \( (2, 2) \) does not

\implies \text{not closed under scaling}
Example. $V = \mathbb{R}^3$.

- The plane $z = 0$ is a subspace of $\mathbb{R}^3$.
- The plane $z = 1$ is not a subspace of $\mathbb{R}^3$.
- The line $t(1, 1, 0), t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ and a subspace of the plane $z = 0$.
- The line $(1, 1, 1) + t(1, -1, 0), t \in \mathbb{R}$ is not a subspace of $\mathbb{R}^3$ as it lies in the plane $x + y + z = 3$, which does not contain $0$.
- In general, a straight line or a plane in $\mathbb{R}^3$ is a subspace if and only if it passes through the origin.
System of linear equations:

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  \cdots \cdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

Any solution \((x_1, x_2, \ldots, x_n)\) is an element of \(\mathbb{R}^n\).

**Theorem** The solution set of the system is a subspace of \(\mathbb{R}^n\) if and only if all \(b_i = 0\).
**Theorem**  The solution set of a system of linear equations in $n$ variables is a subspace of $\mathbb{R}^n$ if and only if all equations are homogeneous.

*Proof:* “only if”: the zero vector $0 = (0, 0, \ldots, 0)$ is a solution only if all equations are homogeneous.

“if”: a system of homogeneous linear equations is equivalent to a matrix equation $Ax = 0$.

$A0 = 0 \implies 0$ is a solution $\implies$ solution set is not empty.

If $Ax = 0$ and $Ay = 0$ then $A(x + y) = Ax + Ay = 0$.

If $Ax = 0$ then $A(rx) = r(Ax) = 0$. 

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- diagonal matrices: $b = c = 0$
- upper triangular matrices: $c = 0$
- lower triangular matrices: $b = 0$
- symmetric matrices ($A^T = A$): $b = c$
- anti-symmetric (or skew-symmetric) matrices ($A^T = -A$): $a = d = 0, \ c = -b$
- matrices with zero trace: $a + d = 0$

(trace = the sum of diagonal entries)

- matrices with zero determinant, $ad - bc = 0$, do not form a subspace: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.