

MATH 304
Linear Algebra

Lecture 16:
Basis and dimension.

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Equivalently, a subset $S \subset V$ is a basis for V if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \dots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \dots, r_k \in \mathbb{R}$.

Remark on uniqueness. Expansions $\mathbf{v} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3$ and $\mathbf{v} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3 + 0\mathbf{v}_4$ are considered the same.

Examples. • Standard basis for \mathbb{R}^n :

$\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0), \dots$,
 $\mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

• Polynomials $1, x, x^2, \dots, x^{n-1}$ form a basis for $\mathcal{P}_n = \{a_0 + a_1x + \dots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis for \mathcal{P} , the space of all polynomials.

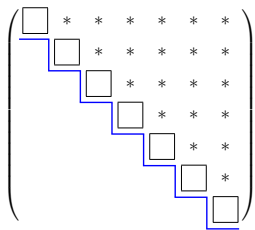
Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$.

The vector equation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{v}$ is equivalent to the matrix equation $A\mathbf{x} = \mathbf{v}$, where

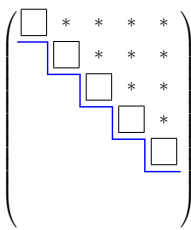
$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \quad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

That is, A is the $n \times k$ matrix such that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are consecutive columns of A .

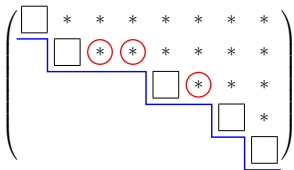
- *Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ span \mathbb{R}^n if the row echelon form of A has no zero rows.*
- *Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent if the row echelon form of A has a leading entry in each column (no free variables).*



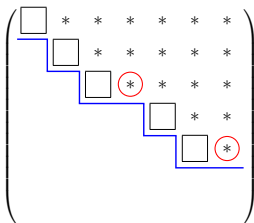
spanning
linear independence



no spanning
linear independence



spanning
no linear independence



no spanning
no linear independence

Bases for \mathbb{R}^n

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors in \mathbb{R}^n .

Theorem 1 If $k < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ do not span \mathbb{R}^n .

Theorem 2 If $k > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly dependent.

Theorem 3 If $k = n$ then the following conditions are equivalent:

- (i) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for \mathbb{R}^n ;
- (ii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for \mathbb{R}^n ;
- (iii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in \mathbb{R}^3 .

Vectors \mathbf{v}_1 and \mathbf{v}_2 are linearly independent (as they are not parallel), but they do not span \mathbb{R}^3 .

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-2) = 2 \neq 0.$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 .

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span \mathbb{R}^3 (because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ already span \mathbb{R}^3), but they are linearly dependent.

Dimension

Theorem 1 Any vector space has a basis.

Theorem 2 If a vector space V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition. The **dimension** of a vector space V , denoted $\dim V$, is the number of elements in any of its bases.

Examples. • $\dim \mathbb{R}^n = n$

• $\mathcal{M}_{2,2}(\mathbb{R})$: the space of 2×2 matrices
 $\dim \mathcal{M}_{2,2}(\mathbb{R}) = 4$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices
 $\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

• \mathcal{P}_n : polynomials of degree less than n
 $\dim \mathcal{P}_n = n$

• \mathcal{P} : the space of all polynomials
 $\dim \mathcal{P} = \infty$

• $\{\mathbf{0}\}$: the trivial vector space
 $\dim \{\mathbf{0}\} = 0$

Problem. Find the dimension of the plane $x + 2z = 0$ in \mathbb{R}^3 .

The general solution of the equation $x + 2z = 0$ is

$$\begin{cases} x = -2s \\ y = t \\ z = s \end{cases} \quad (t, s \in \mathbb{R})$$

That is, $(x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)$.

Hence the plane is the span of vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$. These vectors are linearly independent as they are not parallel.

Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis so that the dimension of the plane is 2.

How to find a basis?

Theorem Let S be a subset of a vector space V . Then the following conditions are equivalent:

- (i) S is a linearly independent spanning set for V , i.e., a basis;
- (ii) S is a minimal spanning set for V ;
- (iii) S is a maximal linearly independent subset of V .

“Minimal spanning set” means “remove any element from this set, and it is no longer a spanning set”.

“Maximal linearly independent subset” means “add any element of V to this set, and it will become linearly dependent”.

Theorem Let V be a vector space. Then

- (i) any spanning set for V can be reduced to a minimal spanning set;
- (ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

Equivalently, any spanning set contains a basis, while any linearly independent set is contained in a basis.

Corollary A vector space is finite-dimensional if and only if it is spanned by a finite set.

How to find a basis?

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis.

Proposition Let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ be a spanning set for a vector space V . If \mathbf{v}_0 is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then $\mathbf{v}_1, \dots, \mathbf{v}_k$ is also a spanning set for V .

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \dots + r_k\mathbf{v}_k$, then

$$\begin{aligned} t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k &= \\ &= (t_0r_1 + t_1)\mathbf{v}_1 + \dots + (t_0r_k + t_k)\mathbf{v}_k. \end{aligned}$$

How to find a basis?

Approach 2. Build a maximal linearly independent set adding one vector at a time.

If the vector space V is trivial, it has the empty basis.

If $V \neq \{\mathbf{0}\}$, pick any vector $\mathbf{v}_1 \neq \mathbf{0}$.

If \mathbf{v}_1 spans V , it is a basis. Otherwise pick any vector $\mathbf{v}_2 \in V$ that is not in the span of \mathbf{v}_1 .

If \mathbf{v}_1 and \mathbf{v}_2 span V , they constitute a basis.

Otherwise pick any vector $\mathbf{v}_3 \in V$ that is not in the span of \mathbf{v}_1 and \mathbf{v}_2 .

And so on...

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for \mathbb{R}^3 .

Our task is to find a vector \mathbf{v}_3 that is not a linear combination of \mathbf{v}_1 and \mathbf{v}_2 .

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for \mathbb{R}^3 .

Hint 1. \mathbf{v}_1 and \mathbf{v}_2 span the plane $x + 2z = 0$.

The vector $\mathbf{v}_3 = (1, 1, 1)$ does not lie in the plane $x + 2z = 0$, hence it is not a linear combination of \mathbf{v}_1 and \mathbf{v}_2 . Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 .

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for \mathbb{R}^3 .

Our task is to find a vector \mathbf{v}_3 that is not a linear combination of \mathbf{v}_1 and \mathbf{v}_2 . Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for \mathbb{R}^3 .

Hint 2. At least one of vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ is a desired one.

Let us check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ are two bases for \mathbb{R}^3 :

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \quad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$