

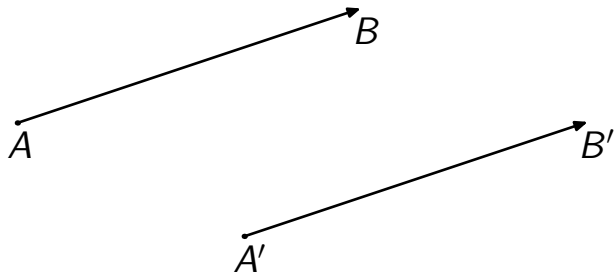
MATH 304

Linear Algebra

Lecture 23:

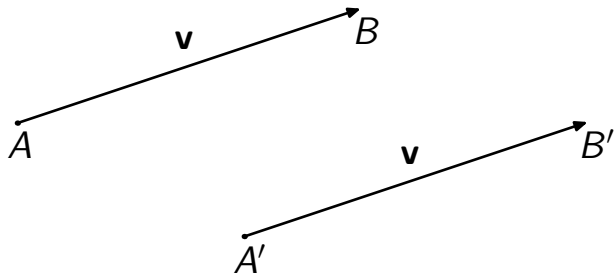
**Euclidean structure in \mathbb{R}^n .
Orthogonality.**

Vectors: geometric approach



- A vector is represented by a directed segment.
- Directed segment is drawn as an arrow.
- Different arrows represent the same vector if they are of the same length and direction.

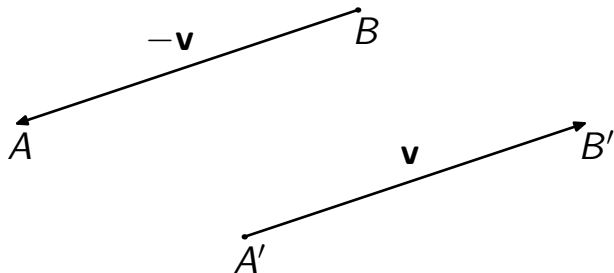
Vectors: geometric approach



\overrightarrow{AB} denotes the vector represented by the arrow with tip at B and tail at A .

\overrightarrow{AA} is called the *zero vector* and denoted $\mathbf{0}$.

Vectors: geometric approach

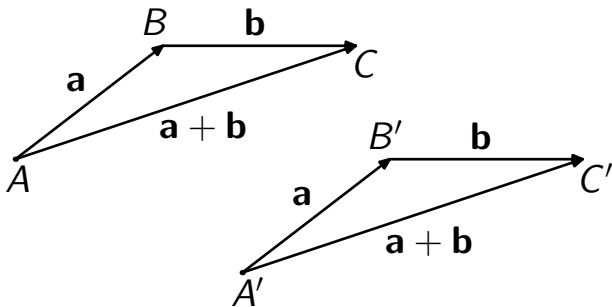


If $\mathbf{v} = \overrightarrow{AB}$ then \overrightarrow{BA} is called the *negative vector* of \mathbf{v} and denoted $-\mathbf{v}$.

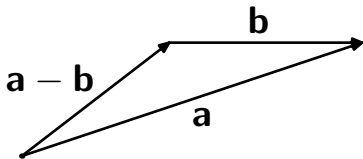
Linear structure: vector addition

Given vectors \mathbf{a} and \mathbf{b} , their *sum* $\mathbf{a} + \mathbf{b}$ is defined by the rule $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

That is, choose points A, B, C so that $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$. Then $\mathbf{a} + \mathbf{b} = \overrightarrow{AC}$.

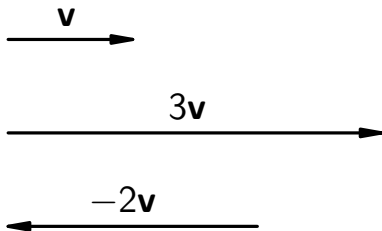


The *difference* of the two vectors is defined as
 $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.



Linear structure: scalar multiplication

Let \mathbf{v} be a vector and $r \in \mathbb{R}$. By definition, $r\mathbf{v}$ is a vector whose magnitude is $|r|$ times the magnitude of \mathbf{v} . The direction of $r\mathbf{v}$ coincides with that of \mathbf{v} if $r > 0$. If $r < 0$ then the directions of $r\mathbf{v}$ and \mathbf{v} are opposite.



Beyond linearity: length of a vector

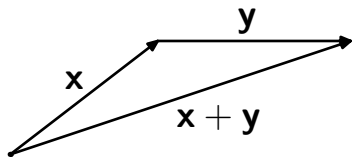
The **length** (or the **magnitude**) of a vector \overrightarrow{AB} is the length of the representing segment AB . The length of a vector \mathbf{v} is denoted $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

Properties of vector length:

$$|\mathbf{x}| \geq 0, \quad |\mathbf{x}| = 0 \text{ only if } \mathbf{x} = \mathbf{0} \quad (\text{positivity})$$

$$|r\mathbf{x}| = |r| |\mathbf{x}| \quad (\text{homogeneity})$$

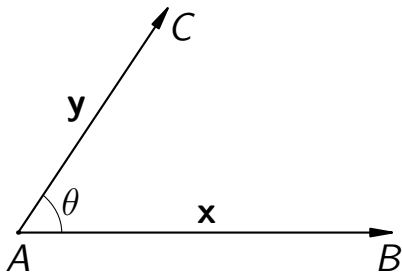
$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}| \quad (\text{triangle inequality})$$

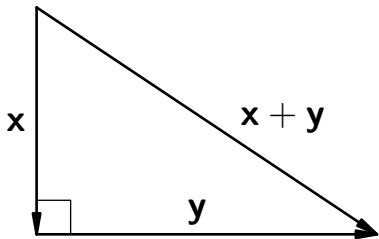


Beyond linearity: angle between vectors

Given nonzero vectors \mathbf{x} and \mathbf{y} , let A , B , and C be points such that $\overrightarrow{AB} = \mathbf{x}$ and $\overrightarrow{AC} = \mathbf{y}$. Then $\angle BAC$ is called the **angle** between \mathbf{x} and \mathbf{y} .

The vectors \mathbf{x} and \mathbf{y} are called **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if the angle between them equals 90° .





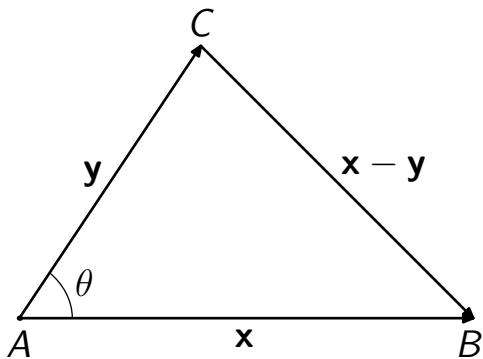
Pythagorean Theorem:

$$\mathbf{x} \perp \mathbf{y} \implies |\mathbf{x} + \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

3-dimensional Pythagorean Theorem:

If vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are pairwise orthogonal then

$$|\mathbf{x} + \mathbf{y} + \mathbf{z}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2$$



Law of cosines:

$$|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}||\mathbf{y}|\cos\theta$$

Beyond linearity: dot product

The **dot product** of vectors \mathbf{x} and \mathbf{y} is

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}| |\mathbf{y}| \cos \theta,$$

where θ is the angle between \mathbf{x} and \mathbf{y} .

The dot product is also called the **scalar product**.

Alternative notation: (\mathbf{x}, \mathbf{y}) or $\langle \mathbf{x}, \mathbf{y} \rangle$.

The vectors \mathbf{x} and \mathbf{y} are orthogonal if and only if

$$\mathbf{x} \cdot \mathbf{y} = 0.$$

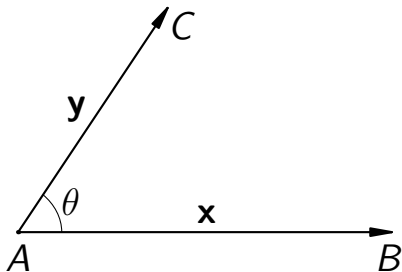
Relations between lengths and dot products:

- $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$
- $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|$
- $|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2 \mathbf{x} \cdot \mathbf{y}$

Euclidean structure

Euclidean structure includes:

- length of a vector: $|\mathbf{x}|$,
- angle between vectors: θ ,
- dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}| |\mathbf{y}| \cos \theta$.



Vectors: algebraic approach

An n -dimensional coordinate vector is an element of \mathbb{R}^n , i.e., an ordered n -tuple (x_1, x_2, \dots, x_n) of real numbers.

Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be vectors, and $r \in \mathbb{R}$ be a scalar. Then, by definition,

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n),$$

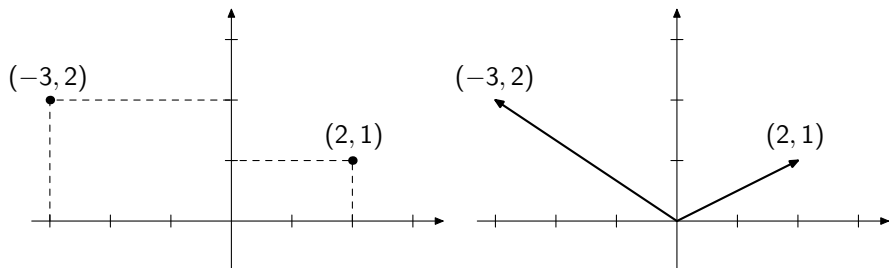
$$r\mathbf{a} = (ra_1, ra_2, \dots, ra_n),$$

$$\mathbf{0} = (0, 0, \dots, 0),$$

$$-\mathbf{b} = (-b_1, -b_2, \dots, -b_n),$$

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \dots, a_n - b_n).$$

Cartesian coordinates: geometric meets algebraic



Once we specify an *origin* O , each point A is associated a *position vector* \overrightarrow{OA} . Conversely, every vector has a unique representative with tail at O .

Cartesian coordinates allow us to identify a line, a plane, and space with \mathbb{R} , \mathbb{R}^2 , and \mathbb{R}^3 , respectively.

Length and distance

Definition. The **length** of a vector

$\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}.$$

The **distance** between vectors/points \mathbf{x} and \mathbf{y} is

$$\|\mathbf{y} - \mathbf{x}\|.$$

Properties of length:

$$\|\mathbf{x}\| \geq 0, \quad \|\mathbf{x}\| = 0 \text{ only if } \mathbf{x} = \mathbf{0} \quad (\text{positivity})$$

$$\|r\mathbf{x}\| = |r| \|\mathbf{x}\| \quad (\text{homogeneity})$$

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \quad (\text{triangle inequality})$$

Scalar product

Definition. The **scalar product** of vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{k=1}^n x_ky_k.$$

Properties of scalar product:

$$\mathbf{x} \cdot \mathbf{x} \geq 0, \quad \mathbf{x} \cdot \mathbf{x} = 0 \text{ only if } \mathbf{x} = \mathbf{0} \quad (\text{positivity})$$

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x} \quad (\text{symmetry})$$

$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z} \quad (\text{distributive law})$$

$$(r\mathbf{x}) \cdot \mathbf{y} = r(\mathbf{x} \cdot \mathbf{y}) \quad (\text{homogeneity})$$

Relations between lengths and scalar products:

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$$

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\| \quad (\text{Cauchy-Schwarz inequality})$$

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x} \cdot \mathbf{y}$$

By the Cauchy-Schwarz inequality, for any nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$\cos \theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|} \quad \text{for some } 0 \leq \theta \leq \pi.$$

θ is called the **angle** between the vectors \mathbf{x} and \mathbf{y} .

The vectors \mathbf{x} and \mathbf{y} are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\mathbf{x} \cdot \mathbf{y} = 0$ (i.e., if $\theta = 90^\circ$).

Problem. Find the angle θ between vectors $\mathbf{x} = (2, -1)$ and $\mathbf{y} = (3, 1)$.

$$\mathbf{x} \cdot \mathbf{y} = 5, \quad \|\mathbf{x}\| = \sqrt{5}, \quad \|\mathbf{y}\| = \sqrt{10}.$$

$$\cos \theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|} = \frac{5}{\sqrt{5} \sqrt{10}} = \frac{1}{\sqrt{2}} \implies \theta = 45^\circ$$

Problem. Find the angle ϕ between vectors $\mathbf{v} = (-2, 1, 3)$ and $\mathbf{w} = (4, 5, 1)$.

$$\mathbf{v} \cdot \mathbf{w} = 0 \implies \mathbf{v} \perp \mathbf{w} \implies \phi = 90^\circ$$

Orthogonality

Definition 1. Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\mathbf{x} \cdot \mathbf{y} = 0$.

Definition 2. A vector $\mathbf{x} \in \mathbb{R}^n$ is said to be **orthogonal** to a nonempty set $Y \subset \mathbb{R}^n$ (denoted $\mathbf{x} \perp Y$) if $\mathbf{x} \cdot \mathbf{y} = 0$ for any $\mathbf{y} \in Y$.

Definition 3. Nonempty sets $X, Y \subset \mathbb{R}^n$ are said to be **orthogonal** (denoted $X \perp Y$) if $\mathbf{x} \cdot \mathbf{y} = 0$ for any $\mathbf{x} \in X$ and $\mathbf{y} \in Y$.

Examples in \mathbb{R}^3 . • The line $x = y = 0$ is orthogonal to the line $y = z = 0$.

Indeed, if $\mathbf{v} = (0, 0, z)$ and $\mathbf{w} = (x, 0, 0)$ then $\mathbf{v} \cdot \mathbf{w} = 0$.

• The line $x = y = 0$ is orthogonal to the plane $z = 0$.

Indeed, if $\mathbf{v} = (0, 0, z)$ and $\mathbf{w} = (x, y, 0)$ then $\mathbf{v} \cdot \mathbf{w} = 0$.

• The line $x = y = 0$ is not orthogonal to the plane $z = 1$.

The vector $\mathbf{v} = (0, 0, 1)$ belongs to both the line and the plane, and $\mathbf{v} \cdot \mathbf{v} = 1 \neq 0$.

• The plane $z = 0$ is not orthogonal to the plane $y = 0$.

The vector $\mathbf{v} = (1, 0, 0)$ belongs to both planes and $\mathbf{v} \cdot \mathbf{v} = 1 \neq 0$.