MATH 304

Linear Algebra

Lecture 37: Rotations in space.

Orthogonal matrices

Definition. A square matrix A is called **orthogonal** if $AA^T = A^TA = I$, i.e., $A^T = A^{-1}$.

Theorem 1 If A is an $n \times n$ orthogonal matrix, then
(i) columns of A form an orthonormal basis for $\mathbb{R}^n$;
(ii) rows of A also form an orthonormal basis for $\mathbb{R}^n$.

Proof: Entries of the matrix A^TA are dot products of columns of A. Entries of AA^T are dot products of rows of A.

Theorem 2 If A is an $n \times n$ orthogonal matrix, then **(i)** A is diagonalizable in the complexified vector space $\mathbb{C}^n$; **(ii)** all eigenvalues λ of A satisfy $|\lambda| = 1$.

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

- $A_\phi A_\psi = A_{\phi+\psi}$
- $A_\phi^{-1} = A_{-\phi} = A_\phi^T$
- A_ϕ is orthogonal
- $\det(A_\phi - \lambda I) = (\cos\phi - \lambda)^2 + \sin^2\phi$.
- Eigenvalues: $\lambda_1 = \cos\phi + i\sin\phi = e^{i\phi}$,

$\lambda_2 = \cos\phi - i\sin\phi = e^{-i\phi}$.

- Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$,

$\mathbf{v}_2 = (1, i)$.

- Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{C}^2$.

Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix.

Theorem The following conditions are equivalent:
(i) $|L(\mathbf{x})| = |\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$;
(ii) $L(\mathbf{x}) \cdot L(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
(iii) the matrix A is orthogonal.

Definition. A transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry** (or a **rigid motion**) if it preserves distances between points: $|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$.

Theorem Any isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ can be represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and A is an orthogonal matrix.

Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ orthogonal matrix.

Theorem There exists an orthonormal basis for $\mathbb{R}^n$ such that the matrix of L relative to this basis has a diagonal block structure

$$\begin{pmatrix} D_{\pm 1} & O & \dots & O \\ O & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & R_k \end{pmatrix},$$

where $D_{\pm 1}$ is a diagonal matrix whose diagonal entries are equal to 1 or -1, and

$$R_j = \begin{pmatrix} \cos \phi_j & -\sin \phi_j \\ \sin \phi_j & \cos \phi_j \end{pmatrix}, \quad \phi_j \in \mathbb{R}.$$

Classification of 2×2 orthogonal matrices:

	$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
	rotation about the origin	reflection in a line
Determinant:	1	-1
Eigenvalues:	$e^{i\phi}$ and $e^{-i\phi}$	-1 and 1

Classification of 3×3 orthogonal matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

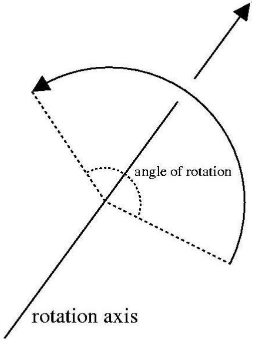
$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

$A =$ rotation about a line; $B =$ reflection in a plane; $C =$ rotation about a line combined with reflection in the orthogonal plane.

$\det A = 1$, $\det B = \det C = -1$.

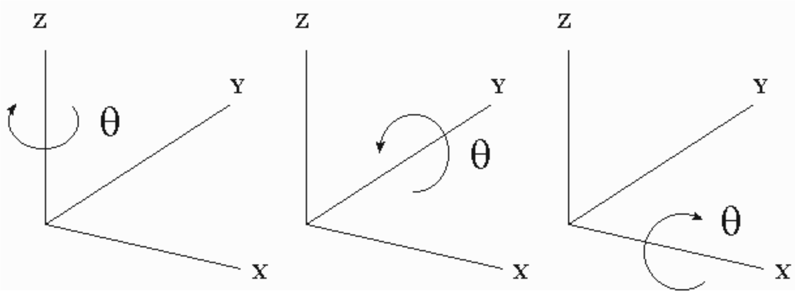
A has eigenvalues 1, $e^{i\phi}$, $e^{-i\phi}$. B has eigenvalues -1, 1, 1. C has eigenvalues -1, $e^{i\phi}$, $e^{-i\phi}$.

Rotations in space



If the axis of rotation is oriented, we can say about *clockwise* or *counterclockwise* rotations (with respect to the view from the positive semi-axis).

Clockwise rotations about coordinate axes



$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

Problem. Find the matrix of the rotation by 90^{o} about the line spanned by the vector $\mathbf{a} = (1, 2, 2)$. The rotation is assumed to be counterclockwise when looking from the tip of **a**.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$ is the matrix of (counterclockwise) rotation by 90^{o} about the x-axis.

We need to find an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $\mathbf{v}_1$ points in the same direction as **a**. Also, the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ should obey the same hand rule as the standard basis. Then B will be the matrix of the given rotation relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let U denote the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (columns of U are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$). Then the desired matrix is $A = UBU^{-1}$.

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is going to be an orthonormal basis, the matrix U will be orthogonal. Then $U^{-1} = U^T$ and $A = UBU^T$.

Remark. The basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the same hand rule as the standard basis if and only if $\det U > 0$.

Hint. Vectors $\mathbf{a} = (1, 2, 2)$, $\mathbf{b} = (-2, -1, 2)$, and $\mathbf{c} = (2, -2, 1)$ are orthogonal.

We have $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 3$, hence $\mathbf{v}_1 = \frac{1}{3}\mathbf{a}$, $\mathbf{v}_2 = \frac{1}{3}\mathbf{b}$, $\mathbf{v}_3 = \frac{1}{3}\mathbf{c}$ is an orthonormal basis.

Transition matrix: $U = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$.

$$\det U = \frac{1}{27}\begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = \frac{1}{27} \cdot 27 = 1.$$

In the case $\det U = -1$, we would change $\mathbf{v}_3$ to $-\mathbf{v}_3$, or change $\mathbf{v}_2$ to $-\mathbf{v}_2$, or interchange $\mathbf{v}_2$ and $\mathbf{v}_3$.

$$A = UBU^T$$
$$= \tfrac{1}{3}\begin{pmatrix}1 & -2 & 2\\ 2 & -1 & -2\\ 2 & 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\cdot\tfrac{1}{3}\begin{pmatrix}1 & 2 & 2\\ -2 & -1 & 2\\ 2 & -2 & 1\end{pmatrix}$$
$$= \tfrac{1}{9}\begin{pmatrix}1 & 2 & 2\\ 2 & -2 & 1\\ 2 & 1 & -2\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ -2 & -1 & 2\\ 2 & -2 & 1\end{pmatrix}$$
$$= \tfrac{1}{9}\begin{pmatrix}1 & -4 & 8\\ 8 & 4 & 1\\ -4 & 7 & 4\end{pmatrix}.$$

$U = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$ is an orthogonal matrix.

$\det U = 1 \implies U$ is a rotation matrix.

Problem. (a) Find the axis of the rotation.
(b) Find the angle of the rotation.

The axis is the set of points $\mathbf{x} \in \mathbb{R}^n$ such that $U\mathbf{x} = \mathbf{x} \iff (U - I)\mathbf{x} = \mathbf{0}$. To find the axis, we apply row reduction to the matrix

$$3(U - I) = 3U - 3I = \begin{pmatrix} -2 & -2 & 2 \\ 2 & -4 & -2 \\ 2 & 2 & -2 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & -2 & 2 \\ 2 & -4 & -2 \\ 2 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 2 & -4 & -2 \\ 2 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -6 & 0 \\ 2 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $U\mathbf{x} = \mathbf{x} \iff \begin{cases} x - z = 0, \\ y = 0. \end{cases}$

The general solution is $x = t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$.

$\implies$ $\mathbf{d} = (1, 0, 1)$ is the direction of the axis.

$$U = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

Let ϕ be the angle of rotation. Then the eigenvalues of U are 1, $e^{i\phi}$, and $e^{-i\phi}$. Therefore

$$\det(U - \lambda I) = (1-\lambda)(e^{i\phi} - \lambda)(e^{-i\phi} - \lambda).$$

Besides, $\det(U - \lambda I) = -\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$, where $c_1 = \operatorname{tr} U$ (the sum of diagonal entries). It follows that

$$\operatorname{tr} U = 1 + e^{i\phi} + e^{-i\phi} = 1 + 2\cos\phi.$$

$$\operatorname{tr} U = 1/3 \implies \cos\phi = -1/3 \implies \phi \approx 109.47^{\mathrm{o}}$$