Sample problems for Test 1: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Let $p(x) = ax^2 + bx + c$. Then $p(1) = a + b + c$, $p(2) = 4a + 2b + c$, and $p(3) = 9a + 3b + c$. The coefficients a, b, and c have to be chosen so that

$$\begin{cases} a + b + c = 1, \\ 4a + 2b + c = 3, \\ 9a + 3b + c = 7. \end{cases}$$

We solve this system of linear equations using elementary operations:

$$\begin{cases} a + b + c = 1 \\ 4a + 2b + c = 3 \\ 9a + 3b + c = 7 \end{cases} \iff \begin{cases} a + b + c = 1 \\ 3a + b = 2 \\ 9a + 3b + c = 7 \end{cases} \iff \begin{cases} a + b + c = 1 \\ 3a + b = 2 \\ 8a + 2b = 6 \end{cases} \iff \begin{cases} a + b + c = 1 \\ 3a + b = 2 \\ 4a + b = 3 \end{cases}$$

$$\iff \begin{cases} a + b + c = 1 \\ 3a + b = 2 \\ a = 1 \end{cases} \iff \begin{cases} a + b + c = 1 \\ b = -1 \\ a = 1 \end{cases} \iff \begin{cases} c = 1 \\ b = -1 \\ a = 1 \end{cases}$$

Thus the desired polynomial is $p(x) = x^2 - x + 1$.

Problem 2 (25 pts.) Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

First let us subtract 2 times the fourth column of A from the first column:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Now the determinant can be easily expanded by the fourth row:

$$\begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix}.$$

The 3×3 determinant is easily expanded by the third row:

$$\begin{vmatrix} -1 & -2 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix} = (-1) \begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix}.$$

Thus

$$\det A = -\begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix} = -1.$$

Another way to evaluate $\det A$ is to reduce the matrix A to the identity matrix using elementary row operations (see below). This requires much more work but we are going to do it anyway, to find the inverse of A.

(ii) Find the inverse matrix A^{-1}.

First we merge the matrix A with the identity matrix into one 4×8 matrix

$$(A\,|\,I) = \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract 2 times the first row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Subtract 2 times the first row from the third row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Subtract 2 times the first row from the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right).$$

Subtract 2 times the fourth row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right).$$

Subtract the fourth row from the third row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right).$$

Add 4 times the second row to the fourth row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array}\right).$$

Add 32 times the third row to the fourth row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right).$$

Add 10 times the third row to the second row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right).$$

Add the fourth row to the first row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right).$$

Add 4 times the third row to the first row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right).$$

Subtract 2 times the second row from the first row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right).$$

Multiply the second, the third, and the fourth rows by -1:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right).$$

Finally the left part of our 4×8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}.$$

As a byproduct, we can evaluate the determinant of A. We have transformed A into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by -1. It follows that $\det I = (-1)^3 \det A$. Hence $\det A = -\det I = -1$.

Problem 3 (20 pts.) Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, a subspace must not be empty.

It is easy to see that each of the sets S_1, S_2, S_3, and S_4 contains the zero vector $(0, 0, 0)$ and all these sets are closed under scalar multiplication.

The set S_1 is the union of three planes $x = 0$, $y = 0$, and $z = 0$. It is not closed under addition as the following example shows: $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$.

S_2 is a plane passing through the origin. Obviously, it is closed under addition.

The condition $y^2 + z^2 = 0$ is equivalent to $y = z = 0$. Hence S_3 is a line passing through the origin. It is closed under addition.

Since $y^2 - z^2 = (y - z)(y + z)$, the set S_4 is the union of two planes $y - z = 0$ and $y + z = 0$. The following example shows that S_4 is not closed under addition: $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$.

Thus S_2 and S_3 are subspaces of $\mathbb{R}^3$ while S_1 and S_4 are not.

Problem 4 (30 pts.) Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(i) Find the rank and the nullity of the matrix B.

The rank (dimension of the row space) and the nullity (dimension of the nullspace) of a matrix are preserved under elementary row operations. We apply such operations to convert the matrix B into row echelon form.

First interchange the first row with the second row:

$$\begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}.$$

Add 3 times the first row to the third row, then subtract 2 times the first row from the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Multiply the second row by -1:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Add the fourth row to the third row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Add 3 times the second row to the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix}.$$

Add 16 times the third row to the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now that the matrix is in row echelon form, its rank equals the number of nonzero rows, which is 3. Since

$$(\text{rank of } B) + (\text{nullity of } B) = (\text{the number of columns of } B) = 4,$$

it follows that the nullity of B equals 1.

(ii) Find a basis for the row space of B, then extend this basis to a basis for $\mathbb{R}^4$.

The row space of a matrix is invariant under elementary row operations. Therefore the row space of the matrix B is the same as the row space of its row echelon form

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The nonzero rows of the latter are linearly independent so that they form a basis for its row space. Hence the vectors $\mathbf{v}_1 = (1, 1, 2, -1)$, $\mathbf{v}_2 = (0, 1, -4, -1)$, and $\mathbf{v}_3 = (0, 0, 1, 0)$ form a basis for the row space of B.

To extend the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to a basis for $\mathbb{R}^4$, we need a vector $\mathbf{v}_4 \in \mathbb{R}^4$ that is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. It is known that at least one of the vectors $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, and $\mathbf{e}_4 = (0, 0, 0, 1)$ can be chosen as $\mathbf{v}_4$. In particular, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_4$ form a basis for $\mathbb{R}^4$. This follows from the fact that the 4×4 matrix whose rows are these vectors is not singular:

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

(iii) Find a basis for the nullspace of B.

The nullspace of B is the solution set of the system of linear homogeneous equations with B as the coefficient matrix. To solve the system, we convert the matrix B to reduced row echelon form. The row echelon form of B has been obtained earlier:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Add 4 times the third row to the second row, then subtract 2 times the third row from the first row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Subtract the second row from the first row:

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have obtained the reduced row echelon form of the matrix B. Its nullspace is the same as the nullspace of B. Hence a vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ belongs to the nullspace of B if and only if

$$\begin{cases} x_1 = 0, \\ x_2 - x_4 = 0, \\ x_3 = 0 \end{cases} \iff \begin{cases} x_1 = 0, \\ x_2 = x_4, \\ x_3 = 0. \end{cases}$$

The general solution of this system is $(x_1, x_2, x_3, x_4) = (0, t, 0, t) = t(0, 1, 0, 1)$, $t \in \mathbb{R}$. Thus the nullspace of the matrix B is spanned by the vector $(0, 1, 0, 1)$. This vector forms a basis for the nullspace.

Bonus Problem 5 (15 pts.) Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Suppose that $af_1(x) + bf_2(x) + cf_3(x) = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Differentiating the identity $af_1(x) + bf_2(x) + cf_3(x) = 0$ four times, we obtain four more identities:

$$\begin{aligned} ax + bxe^x + ce^{-x} &= 0, \\ a + be^x + bxe^x - ce^{-x} &= 0, \\ 2be^x + bxe^x + ce^{-x} &= 0, \\ 3be^x + bxe^x - ce^{-x} &= 0, \\ 4be^x + bxe^x + ce^{-x} &= 0. \end{aligned}$$

Subtracting the third identity from the fifth one, we obtain $2be^x = 0$, which implies that $b = 0$. Substituting $b = 0$ in the third identity, we obtain $ce^{-x} = 0$, which implies that $c = 0$. Substituting $b = 0$ and $c = 0$ in the second identity, we obtain $a = 0$.

Alternative solution: Suppose that $ax + bxe^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

For any $x \neq 0$ divide both sides of the identity by xe^x:

$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$

Note that $e^{-x} \to 0$ and $x^{-1}e^{-2x} \to 0$ as $x \to +\infty$. Hence the left-hand side approaches b as $x \to +\infty$. It follows that $b = 0$. Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the latter identity by x:

$$a + cx^{-1}e^{-x} = 0.$$

Since $x^{-1}e^{-x} \to 0$ as $x \to +\infty$, the left-hand side approaches a as $x \to +\infty$. It follows that $a = 0$. Then $ce^{-x} = 0$, which implies that $c = 0$.

Bonus Problem 6 (15 pts.) Let V be a finite-dimensional vector space and V_0 be a proper subspace of V (where proper means that $V_0 \neq V$). Prove that $\dim V_0 < \dim V$.

Any linearly independent set in a vector space can be extended to a basis. Since the vector space V is finite dimensional, it does not admit infinitely many linearly independent vectors. Clearly, the same is true for the subspace V_0. It follows that V_0 is also finite-dimensional.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be a basis for V_0. The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent in V since they are linearly independent in V_0. Therefore we can extend this collection of vectors to a basis for V by adding some vectors $\mathbf{w}_1, \dots, \mathbf{w}_m$. As $V_0 \neq V$, we do need to add some vectors, i.e., $m \geq 1$. Thus $\dim V_0 = k$ and $\dim V = k + m > k$.