Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Find a quadratic polynomial p(x) such that p(-1) = p(3) = 6 and p'(2) = p(1).

Problem 2 (20 pts.) Let
$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}$$

(i) Evaluate the determinant of the matrix A.

(ii) Find the inverse matrix A^{-1} .

Problem 3 (20 pts.) Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on \mathbb{R}^3 such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.

(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for \mathbb{R}^3 .

(ii) Find the matrix of the operator L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(iii) Find the matrix of the operator L relative to the standard basis.

Problem 4 (25 pts.) Let
$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
.

(i) Find all eigenvalues of the matrix B.

(ii) Find a basis for \mathbb{R}^3 consisting of eigenvectors of B.

(iii) Find an orthonormal basis for \mathbb{R}^3 consisting of eigenvectors of B.

(iv) Find a diagonal matrix D and an invertible matrix U such that $B = UDU^{-1}$.

Problem 5 (20 pts.) Let V be a subspace of \mathbb{R}^4 spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (2, 0, -1, 1)$, and $\mathbf{x}_3 = (0, 1, 1, 0)$.

(i) Find the distance from the point $\mathbf{y} = (0, 0, 0, 4)$ to the subspace V.

(ii) Find the distance from the point **y** to the orthogonal complement V^{\perp} .

Bonus Problem 6 (15 pts.) Consider a linear operator $K : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$K(\mathbf{x}) = C\mathbf{x}$$
, where $C = \frac{1}{9} \begin{pmatrix} -4 & 7 & 4\\ 1 & -4 & 8\\ 8 & 4 & 1 \end{pmatrix}$.

The operator K is a rotation about an axis.

- (i) Find the axis of rotation.
- (ii) Find the angle of rotation.

Bonus Problem 7 (15 pts.) Let P be a square matrix. Assuming P is diagonalizable, prove that $det(exp P) = e^{trace(P)}$.