MATH 304

Lecture 3:

Linear Algebra

Row echelon form.
Gauss-Jordan reduction.

System of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

Here x_1, x_2, \ldots, x_n are variables and a_{ij}, b_j are constants.

A *solution* of the system is a common solution of all equations in the system.

A system of linear equations can have **one** solution, **infinitely many** solutions, or **no** solution at all.

Gaussian elimination

Solution of a system of linear equations splits into two parts: **(A)** elimination and **(B)** back substitution. Both parts can be done by applying a finite number of **elementary operations**.

Elementary operations for systems of linear equations:

- (1) to multiply an equation by a nonzero scalar;
- (2) to add an equation multiplied by a scalar to another equation;
- (3) to interchange two equations.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix and column vector of the right-hand sides:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

Augmented matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the **augmented matrix**.

Elementary operations for systems of linear equations correspond to **elementary row operations** for augmented matrices:

- (1) to multiply a row by a nonzero scalar;
- (2) to add the *i*th row multiplied by some $r \in \mathbb{R}$ to the *j*th row;
- (3) to interchange two rows.

Remark. Rows are added and multiplied by scalars as vectors (namely, row vectors).

Row echelon form

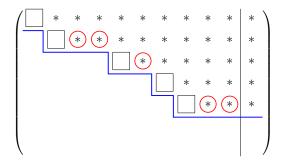
Definition. Leading entry of a matrix is the first nonzero entry in a row.

The goal of the Gaussian elimination is to convert the augmented matrix into **row echelon form**:

- leading entries shift to the right as we go from the first row to the last one;
 - each leading entry is equal to 1.

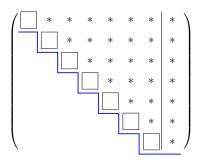
Row echelon form

General augmented matrix in row echelon form:



- leading entries are boxed (all equal to 1);
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a column without a leading entry that corresponds to a free variable.

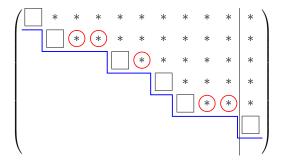
Strict triangular form is a particular case of row echelon form that can occur for systems of n equations in n variables:



- no zero rows;
- no free variables.

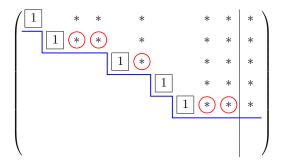
Consistency check

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.



Augmented matrix of an inconsistent system

The goal of the **Gauss-Jordan reduction** is to convert the augmented matrix into **reduced row echelon form**:



- all entries below the staircase line are zero;
- each boxed entry is 1, the other entries in its column are zero;
 - each circle corresponds to a free variable.

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases} \qquad \begin{pmatrix} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{pmatrix}$$

Row echelon form (also strict triangular):

$$\begin{cases} x - y & = 2 \\ y - z = -1 \\ z = 2 \end{cases} \begin{pmatrix} \boxed{1} & -1 & 0 & 2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & \boxed{1} & 2 \end{pmatrix}$$

Reduced row echelon form:

Reduced row echelon form:
$$\begin{cases}
x & = 3 \\
y & = 1 \\
z & = 2
\end{cases}$$

$$\begin{pmatrix}
\boxed{1} & 0 & 0 & 3 \\
0 & \boxed{1} & 0 & 1 \\
0 & 0 & \boxed{1} & 2
\end{pmatrix}$$

Another example.

$$\begin{cases} x + y - 2 \\ y - y - 1 \end{cases}$$

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases} \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 14 \end{pmatrix}$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases} \qquad \begin{pmatrix} \boxed{1} \\ 0 \\ 0 \end{pmatrix}$$

Reduced row echelon form:

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

$$\begin{pmatrix}
1 & 1 & -2 & | & 1 \\
0 & 1 & -1 & | & 3 \\
-1 & 4 & -3 & | & 1
\end{pmatrix}$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 1 \end{cases}$$

$$\begin{pmatrix}
\boxed{1} & 1 & -2 & | & 1 \\
0 & \boxed{1} & -1 & | & 3 \\
0 & 0 & 0 & \boxed{1}
\end{pmatrix}$$

Reduced row echelon form:

$$\begin{cases} x & -z = 0 \\ y - z = 0 \\ 0 = 1 \end{cases}$$

$$\begin{pmatrix}
\boxed{1} & 0 & -1 & 0 \\
0 & \boxed{1} & -1 & 0 \\
0 & 0 & 0 & \boxed{1}
\end{pmatrix}$$

How to solve a system of linear equations

- Order the variables.
- Write down the augmented matrix of the system.
- Convert the matrix to row echelon form.
- Check for consistency.
- Convert the matrix to **reduced row echelon form**.
- Write down the system corresponding to the reduced row echelon form.
- Determine leading and free variables.
- Rewrite the system so that the leading variables are on the left while everything else is on the right.
- Assign parameters to the free variables and write down the general solution in parametric form.

New example.
$$\begin{cases} x_2 + 2x_3 + 3x_4 = 6 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \end{cases}$$

Variables: x_1, x_2, x_3, x_4 .

Augmented matrix:
$$\begin{pmatrix} 0 & 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 4 & 10 \end{pmatrix}$$

To get it into row echelon form, we exchange the two rows:

$$\left(\begin{array}{ccc|c}
1 & 2 & 3 & 4 & 10 \\
0 & 1 & 2 & 3 & 6
\end{array}\right)$$

Consistency check is passed. To convert into reduced row echelon form, add -2 times the 2nd row to the 1st row:

$$\begin{pmatrix}
\boxed{1} & 0 & -1 & -2 & | & -2 \\
0 & \boxed{1} & 2 & 3 & | & 6
\end{pmatrix}$$

The leading variables are x_1 and x_2 ; hence x_3 and x_4 are free variables.

Back to the system:

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

 $(t, s \in \mathbb{R})$

General solution:

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} (t, s \in \mathbb{R})$$
In vector form, $(x_1, x_2, x_3, x_4) = (-2, 6, 0, 0) + t(1, -2, 1, 0) + s(2, -3, 0, 1).$

Example with a parameter.

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases} (a \in \mathbb{R})$$

The system is **homogeneous** (all right-hand sides are zeros). Therefore it is consistent (x = y = z = 0) is a solution).

Augmented matrix:
$$\begin{pmatrix} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{pmatrix}$$

Since the 1st row cannot serve as a pivotal one, we interchange it with the 2nd row:

$$\begin{pmatrix} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{pmatrix}$$
Now we can start the elimination.

First subtract the 1st row from the 3rd row.

First subtract the 1st row from the 3rd row:
$$\begin{pmatrix}
1 & 1 & -2 & 0 \\
0 & 1 & 3 & 0 \\
1 & 2 & a & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & -2 & 0 \\
0 & 1 & 3 & 0 \\
0 & 1 & a+2 & 0
\end{pmatrix}$$

The 2nd row is our new pivotal row.

Subtract the 2nd row from the 3rd row:

$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{pmatrix}$$

At this point row reduction splits into two cases.

Case 1: $a \neq 1$. In this case, multiply the 3rd row by $(a-1)^{-1}$:

$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{pmatrix}$$

The matrix is converted into row echelon form. We proceed towards reduced row echelon form.

Subtract 3 times the 3rd row from the 2nd row:

$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Add 2 times the 3rd row to the 1st row:

$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Finally, subtract the 2nd row from the 1st row:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{pmatrix}$$

Thus x = y = z = 0 is the only solution.

Case 2: a = 1. In this case, the matrix is already in row echelon form:

$$\begin{pmatrix}
\boxed{1} & 1 & -2 & 0 \\
0 & \boxed{1} & 3 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

To get reduced row echelon form, subtract the 2nd row from the 1st row:

$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \boxed{1} & 0 & -5 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

z is a free variable.

$$\begin{cases} x - 5z = 0 \\ y + 3z = 0 \end{cases} \iff \begin{cases} x = 5z \\ y = -3z \end{cases}$$

System of linear equations:

$$\int y + 3z = 0$$

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases}$$

Solution: If $a \neq 1$ then (x, y, z) = (0, 0, 0); if a = 1 then (x, y, z) = (5t, -3t, t), $t \in \mathbb{R}$.