

MATH 304  
Linear Algebra

**Lecture 14:**  
**Linear independence.**

## Spanning set

Let  $S$  be a subset of a vector space  $V$ .

*Definition.* The **span** of the set  $S$  is the smallest subspace  $W \subset V$  that contains  $S$ . If  $S$  is not empty then  $W = \text{Span}(S)$  consists of all linear combinations  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$  such that  $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$  and  $r_1, \dots, r_k \in \mathbb{R}$ .

We say that the set  $S$  **spans** the subspace  $W$  or that  $S$  is a **spanning set** for  $W$ .

*Remarks.* • If  $S_1$  is a spanning set for a vector space  $V$  and  $S_1 \subset S_2 \subset V$ , then  $S_2$  is also a spanning set for  $V$ .

• If  $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$  is a spanning set for  $V$  and  $\mathbf{v}_0$  is a linear combination of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  then  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is also a spanning set for  $V$ .

## Linear independence

*Definition.* Let  $V$  be a vector space. Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$  are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients  $r_1, \dots, r_k \in \mathbb{R}$  are not all equal to zero. Otherwise vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

A set  $S \subset V$  is **linearly dependent** if one can find some distinct linearly dependent vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  in  $S$ . Otherwise  $S$  is **linearly independent**.

## Examples of linear independence

- Vectors  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$ , and  $\mathbf{e}_3 = (0, 0, 1)$  in  $\mathbb{R}^3$ .

$$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = \mathbf{0} \implies (x, y, z) = \mathbf{0} \\ \implies x = y = z = 0$$

- Matrices  $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  
 $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ , and  $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ .

$$aE_{11} + bE_{12} + cE_{21} + dE_{22} = O \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O \\ \implies a = b = c = d = 0$$

## Examples of linear independence

- Polynomials  $1, x, x^2, \dots, x^n$ .

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0 \text{ identically} \\ \implies a_i = 0 \text{ for } 0 \leq i \leq n$$

- The infinite set  $\{1, x, x^2, \dots, x^n, \dots\}$ .

- Polynomials  $p_1(x) = 1$ ,  $p_2(x) = x - 1$ , and  $p_3(x) = (x - 1)^2$ .

$$a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = a_1 + a_2(x - 1) + a_3(x - 1)^2 = \\ = (a_1 - a_2 + a_3) + (a_2 - 2a_3)x + a_3x^2.$$

$$\text{Hence } a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = 0 \text{ identically}$$

$$\implies a_1 - a_2 + a_3 = a_2 - 2a_3 = a_3 = 0$$

$$\implies a_1 = a_2 = a_3 = 0$$

**Problem** Let  $\mathbf{v}_1 = (1, 2, 0)$ ,  $\mathbf{v}_2 = (3, 1, 1)$ , and  $\mathbf{v}_3 = (4, -7, 3)$ . Determine whether vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent.

We have to check if there exist  $r_1, r_2, r_3 \in \mathbb{R}$  not all zero such that  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 = \mathbf{0}$ .

This vector equation is equivalent to a system

$$\begin{cases} r_1 + 3r_2 + 4r_3 = 0 \\ 2r_1 + r_2 - 7r_3 = 0 \\ 0r_1 + r_2 + 3r_3 = 0 \end{cases} \quad \left( \begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 1 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array} \right)$$

The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly dependent if and only if the matrix  $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  is singular.

We obtain that  $\det A = 0$ .

**Theorem** The following conditions are equivalent:

- (i) vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are linearly dependent;
- (ii) one of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is a linear combination of the other  $k - 1$  vectors.

*Proof:* (i)  $\implies$  (ii) Suppose that

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where  $r_i \neq 0$  for some  $1 \leq i \leq k$ . Then

$$\mathbf{v}_i = -\frac{r_1}{r_i}\mathbf{v}_1 - \cdots - \frac{r_{i-1}}{r_i}\mathbf{v}_{i-1} - \frac{r_{i+1}}{r_i}\mathbf{v}_{i+1} - \cdots - \frac{r_k}{r_i}\mathbf{v}_k.$$

(ii)  $\implies$  (i) Suppose that

$$\mathbf{v}_i = s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k$$

for some scalars  $s_j$ . Then

$$s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k = \mathbf{0}.$$

**Theorem** Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$  are linearly dependent whenever  $m > n$  (i.e., the number of coordinates is less than the number of vectors).

*Proof:* Let  $\mathbf{v}_j = (a_{1j}, a_{2j}, \dots, a_{nj})$  for  $j = 1, 2, \dots, m$ . Then the vector equality  $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \dots + t_m\mathbf{v}_m = \mathbf{0}$  is equivalent to the system

$$\begin{cases} a_{11}t_1 + a_{12}t_2 + \dots + a_{1m}t_m = 0, \\ a_{21}t_1 + a_{22}t_2 + \dots + a_{2m}t_m = 0, \\ \dots\dots\dots \\ a_{n1}t_1 + a_{n2}t_2 + \dots + a_{nm}t_m = 0. \end{cases}$$

Note that vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  are columns of the matrix  $(a_{ij})$ . The number of leading entries in the row echelon form is at most  $n$ . If  $m > n$  then there are free variables, therefore the zero solution is not unique.



*Example.* Consider vectors  $\mathbf{v}_1 = (1, -1, 1)$ ,  $\mathbf{v}_2 = (1, 0, 0)$ ,  $\mathbf{v}_3 = (1, 1, 1)$ , and  $\mathbf{v}_4 = (1, 2, 4)$  in  $\mathbb{R}^3$ .

Two vectors are linearly dependent if and only if they are parallel. Hence  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent.

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent if and only if the matrix  $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  is invertible.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Therefore  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent.

Four vectors in  $\mathbb{R}^3$  are always linearly dependent.

Thus  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  are linearly dependent.

**Problem.** Let  $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ . Determine whether matrices  $A$ ,  $A^2$ , and  $A^3$  are linearly independent.

We have  $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ ,  $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

The task is to check if there exist  $r_1, r_2, r_3 \in \mathbb{R}$  not all zero such that  $r_1 A + r_2 A^2 + r_3 A^3 = O$ .

This matrix equation is equivalent to a system

$$\begin{cases} -r_1 + 0r_2 + r_3 = 0 \\ r_1 - r_2 + 0r_3 = 0 \\ -r_1 + r_2 + 0r_3 = 0 \\ 0r_1 - r_2 + r_3 = 0 \end{cases} \quad \left( \begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The row echelon form of the augmented matrix shows there is a free variable. Hence the system has a nonzero solution so that the matrices are linearly dependent (one relation is  $A + A^2 + A^3 = O$ ).

## More facts on linear independence

Let  $S_0$  and  $S$  be subsets of a vector space  $V$ .

- If  $S_0 \subset S$  and  $S$  is linearly independent, then so is  $S_0$ .
- If  $S_0 \subset S$  and  $S_0$  is linearly dependent, then so is  $S$ .
- If  $S$  is linearly independent in  $V$  and  $V$  is a subspace of  $W$ , then  $S$  is linearly independent in  $W$ .
- The empty set is linearly independent.
- Any set containing  $\mathbf{0}$  is linearly dependent.
- Two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly dependent if and only if one of them is a scalar multiple the other.
  - Two nonzero vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly dependent if and only if either of them is a scalar multiple the other.
- If  $S_0$  is linearly independent and  $\mathbf{v}_0 \in V \setminus S_0$  then  $S_0 \cup \{\mathbf{v}_0\}$  is linearly independent if and only if  $\mathbf{v}_0 \notin \text{Span}(S_0)$ .