MATH 304 Linear Algebra

Lecture 9: Properties of determinants.

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ii})_{1 \le i, i \le n}$ is denoted det A or

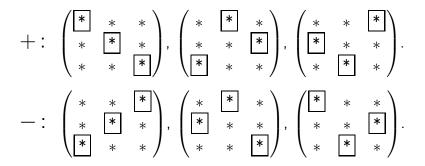
I					
	a_{11}	a_{12}	• • •	a _{1n}	
	a ₂₁	<i>a</i> ₂₂	• • •	a _{2n}	
	÷	÷	•••	÷	•
	a _{n1}	a _{n2}	•••	a _{nn}	

Principal property: det $A \neq 0$ if and only if a system of linear equations with the coefficient matrix A has a unique solution. Equivalently, det $A \neq 0$ if and only if the matrix A is invertible.

Definition in low dimensions

Definition. det (a) = a,
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
 = ad - bc, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \end{vmatrix}$

 $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$



Let us try to find a solution of a general system of 2 linear equations in 2 variables:

$$\begin{cases} a_{11}x + a_{12}y = b_1, \\ a_{21}x + a_{22}y = b_2. \end{cases}$$

Solve the 1st equation for x: $x = (b_1 - a_{12}y)/a_{11}$. Substitute into the 2nd equation:

$$a_{21}(b_1 - a_{12}y)/a_{11} + a_{22}y = b_2.$$

Solve for y: $y = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}.$
Back substitution: $x = (b_1 - a_{12}y)/a_{11} = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}.$

Thus

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

General definition

The general definition of the determinant is quite complicated as there is no simple explicit formula.

There are several approaches to defining determinants.

Approach 1 (original): an explicit (but very complicated) formula.

Approach 2 (axiomatic): we formulate properties that the determinant should have.

Approach 3 (inductive): the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n-1) \times (n-1)$ matrices.

Axiomatic definition

 $\mathcal{M}_{n,n}(\mathbb{R})$: the set of $n \times n$ matrices with real entries.

Theorem There exists a unique function det : $\mathcal{M}_{n,n}(\mathbb{R}) \to \mathbb{R}$ (called the determinant) with the following properties:

(D1) if a row of a matrix is multiplied by a scalar *r*, the determinant is also multiplied by *r*;

(D2) if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;

(D3) if we interchange two rows of a matrix, the determinant changes its sign;

(D4) det I = 1.

Corollary 1 Suppose A is a square matrix and B is obtained from A applying elementary row operations. Then det A = 0 if and only if det B = 0.

Corollary 2 det B = 0 whenever the matrix B has a zero row.

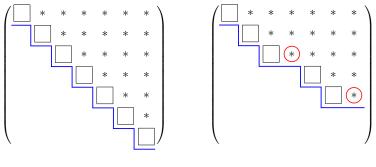
Hint: Multiply the zero row by the zero scalar.

Corollary 3 det A = 0 if and only if the matrix A is not invertible.

Idea of the proof: Let *B* be the reduced row echelon form of *A*. If *A* is invertible then B = I; otherwise *B* has a zero row.

Remark. The same argument proves that properties (D1)-(D4) are enough to evaluate any determinant.

Row echelon form of a square matrix A:



 $\det A \neq 0 \qquad \qquad \det A = 0$

Example.
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
, det $A = ?$

Earlier we have transformed the matrix A into the identity matrix using elementary row operations:

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by -0.5,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by -0.4,
- add -1.5 times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

Example.
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
, det $A = ?$

Earlier we have transformed the matrix A into the identity matrix using elementary row operations.

These included two row multiplications, by -0.5 and by -0.4, and one row exchange.

It follows that

det I = -(-0.5)(-0.4) det A = (-0.2) det A. Hence det A = -5 det I = -5.

Other properties of determinants

• If a matrix A has two identical rows then det A = 0.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

• If a matrix A has two proportional rows then $\det A = 0$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ ra_1 & ra_2 & ra_3 \end{vmatrix} = r \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0.$$

Additive law for rows

• Suppose that matrices X, Y, Z are identical except for the *i*th row and the *i*th row of Z is the sum of the *i*th rows of X and Y.

Then det $Z = \det X + \det Y$.

$$\begin{vmatrix} a_1 + a_1' & a_2 + a_2' & a_3 + a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & a_2' & a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

• Adding a scalar multiple of one row to another row does not change the determinant of a matrix.

$$\begin{vmatrix} a_1 + rb_1 & a_2 + rb_2 & a_3 + rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} rb_1 & rb_2 & rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Definition. A square matrix $A = (a_{ij})$ is called upper triangular if all entries below the main diagonal are zeros: $a_{ij} = 0$ whenever i > j.

• The determinant of an upper triangular matrix is equal to the product of its diagonal entries.

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{array} = a_{11}a_{22}a_{33}.$$

• If $A = \text{diag}(d_1, d_2, \dots, d_n)$ then det $A = d_1 d_2 \dots d_n$. In particular, det I = 1.

Determinant of the transpose

• If A is a square matrix then det $A^T = \det A$.

<i>a</i> ₁	b_1	<i>c</i> ₁		a ₁	a 2	a 3
a ₂	b_2	<i>c</i> ₂	=	b_1	b_2	<i>b</i> ₃
a ₃	<i>b</i> ₃	<i>C</i> 3		<i>c</i> ₁	<i>c</i> ₂	C ₃

As a consequence, for every property of determinants involving rows of a matrix there is an analogous property involving columns of a matrix.

Columns vs. rows

• If one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar.

• Interchanging two columns of a matrix changes the sign of its determinant.

• If a matrix A has two columns proportional then $\det A = 0$.

• Adding a scalar multiple of one column to another does not change the determinant of a matrix.

Submatrices

Definition. Given a matrix A, a $k \times k$ submatrix of A is a matrix obtained by specifying k columns and k rows of A and deleting the other columns and rows.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 10 & 20 & 30 & 40 \\ 3 & 5 & 7 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} * & 2 & * & 4 \\ * & * & * & * \\ * & 5 & * & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 \\ 5 & 9 \end{pmatrix}$$

Row and column expansions

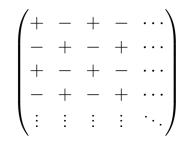
Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the *i*th row and the *j*th column of A.

Theorem For any $1 \le k, m \le n$ we have that

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj},$$

(expansion by k-th row)
 $\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}.$
(expansion by m-th column)

Signs for row/column expansions



Example.
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$
.

Expansion by the 1st row:

$$\begin{pmatrix} \boxed{1} & * & * \\ * & 5 & 6 \\ * & 8 & 9 \end{pmatrix} \begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \begin{pmatrix} * & * & \boxed{3} \\ 4 & 5 & * \\ 7 & 8 & * \end{pmatrix}$$
$$\det A = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$
$$= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0.$$

Example.
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$
.

Expansion by the 2nd column:

$$\begin{pmatrix} * & 2 & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \begin{pmatrix} 1 & * & 3 \\ * & 5 & * \\ 7 & * & 9 \end{pmatrix} \begin{pmatrix} 1 & * & 3 \\ 4 & * & 6 \\ * & 8 & * \end{pmatrix}$$
$$det A = -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$
$$= -2(4 \cdot 9 - 6 \cdot 7) + 5(1 \cdot 9 - 3 \cdot 7) - 8(1 \cdot 6 - 3 \cdot 4) = 0.$$

Example.
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$
.

Subtract the 1st row from the 2nd row and from the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$$

since the last matrix has two proportional rows.