MATH 304

Linear Algebra

Lecture 12:
Vector spaces (continued).
Subspaces of vector spaces.

Abstract vector space

A *vector space* is a set V equipped with two operations, **addition** $V \times V \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y} \in V$ and **scalar multiplication** $\mathbb{R} \times V \ni (r, \mathbf{x}) \mapsto r\mathbf{x} \in V$, that have the following properties:

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$;

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$;

A3. there exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$;

A4. for any $\mathbf{x} \in V$ there exists an element of V, denoted $-\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$;

A5. $r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$ for all $r \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$;

A6. $(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$ for all $r, s \in \mathbb{R}$ and $\mathbf{x} \in V$;

A7. $(rs)\mathbf{x} = r(s\mathbf{x})$ for all $r, s \in \mathbb{R}$ and $\mathbf{x} \in V$;

A8. $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.

Additional properties of vector spaces

- The zero vector is unique.
- For any $\mathbf{x} \in V$, the negative $-\mathbf{x}$ is unique.
- $\mathbf{x} + \mathbf{y} = \mathbf{z} \iff \mathbf{x} = \mathbf{z} - \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.
- $\mathbf{x} + \mathbf{y} = \mathbf{x}' + \mathbf{y} \iff \mathbf{x} = \mathbf{x}'$ for all $\mathbf{x}, \mathbf{x}', \mathbf{y} \in V$.
- $0\mathbf{x} = \mathbf{0}$ for any $\mathbf{x} \in V$.
- $(-1)\mathbf{x} = -\mathbf{x}$ for any $\mathbf{x} \in V$.

Additional properties of vector spaces

- (cancellation law) $\mathbf{x}+\mathbf{y}=\mathbf{x}'+\mathbf{y}$ implies $\mathbf{x}=\mathbf{x}'$ for any $\mathbf{x},\mathbf{x}',\mathbf{y}\in V$.

If $\mathbf{x}+\mathbf{y}=\mathbf{x}'+\mathbf{y}$ then $(\mathbf{x}+\mathbf{y})+(-\mathbf{y})=(\mathbf{x}'+\mathbf{y})+(-\mathbf{y})$. By associativity, $(\mathbf{x}+\mathbf{y})+(-\mathbf{y})=\mathbf{x}+(\mathbf{y}+(-\mathbf{y}))=\mathbf{x}+\mathbf{0}=\mathbf{x}$ and $(\mathbf{x}'+\mathbf{y})+(-\mathbf{y})=\mathbf{x}'+(\mathbf{y}+(-\mathbf{y}))=\mathbf{x}'+\mathbf{0}=\mathbf{x}'$. Hence $\mathbf{x}=\mathbf{x}'$.

- $0\mathbf{x}=\mathbf{0}$ for any $\mathbf{x}\in V$.

Indeed, $0\mathbf{x}+\mathbf{x}=0\mathbf{x}+1\mathbf{x}=(0+1)\mathbf{x}=1\mathbf{x}=\mathbf{x}=\mathbf{0}+\mathbf{x}$. By the cancellation law, $0\mathbf{x}=\mathbf{0}$.

- $(-1)\mathbf{x}=-\mathbf{x}$ for any $\mathbf{x}\in V$.

Indeed, $\mathbf{x}+(-1)\mathbf{x}=(-1)\mathbf{x}+\mathbf{x}=(-1)\mathbf{x}+1\mathbf{x}=(-1+1)\mathbf{x}$ $=0\mathbf{x}=\mathbf{0}$.

Examples of vector spaces

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m \times n$ matrices with real entries
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \ldots)$, $x_i \in \mathbb{R}$
- $\{\mathbf{0}\}$: the trivial vector space
- $F(\mathbb{R})$: the set of all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$
- $C^1(\mathbb{R})$: all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$
- $C^\infty(\mathbb{R})$: all smooth functions $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

Counterexample: dumb scaling

Consider the set $V = \mathbb{R}^n$ with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{x} = \mathbf{0}} \quad \text{for any } \mathbf{x} \in \mathbb{R}^n \text{ and } r \in \mathbb{R}.$$

Properties A1–A4 hold because they do not involve scalar multiplication.

A5. $r \odot (\mathbf{x} + \mathbf{y}) = r \odot \mathbf{x} + r \odot \mathbf{y} \iff \mathbf{0} = \mathbf{0} + \mathbf{0}$

A6. $(r + s) \odot \mathbf{x} = r \odot \mathbf{x} + s \odot \mathbf{x} \iff \mathbf{0} = \mathbf{0} + \mathbf{0}$

A7. $(rs) \odot \mathbf{x} = r \odot (s \odot \mathbf{x}) \iff \mathbf{0} = \mathbf{0}$

A8. $1 \odot \mathbf{x} = \mathbf{x} \iff \mathbf{0} = \mathbf{x}$

A8 is the only property that fails. As a consequence, property A8 does not follow from properties A1–A7.

Counterexample: lazy scaling

Consider the set $V = \mathbb{R}^n$ with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{x} = \mathbf{x}} \quad \text{for any } \mathbf{x} \in \mathbb{R}^n \text{ and } r \in \mathbb{R}.$$

Properties A1–A4 hold because they do not involve scalar multiplication.

A5. $r \odot (\mathbf{x} + \mathbf{y}) = r \odot \mathbf{x} + r \odot \mathbf{y} \iff \mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{y}$

A6. $(r + s) \odot \mathbf{x} = r \odot \mathbf{x} + s \odot \mathbf{x} \iff \mathbf{x} = \mathbf{x} + \mathbf{x}$

A7. $(rs) \odot \mathbf{x} = r \odot (s \odot \mathbf{x}) \iff \mathbf{x} = \mathbf{x}$

A8. $1 \odot \mathbf{x} = \mathbf{x} \iff \mathbf{x} = \mathbf{x}$

The only property that fails is A6.

Weird example

Consider the set $V = \mathbb{R}_+$ of positive numbers with a nonstandard addition and scalar multiplication:

$$\boxed{x \oplus y = xy} \quad \text{for any } x, y \in \mathbb{R}_+,$$
$$\boxed{r \odot x = x^r} \quad \text{for any } x \in \mathbb{R}_+ \text{ and } r \in \mathbb{R}.$$

A1. $x \oplus y = y \oplus x \iff xy = yx$

A2. $(x \oplus y) \oplus z = x \oplus (y \oplus z) \iff (xy)z = x(yz)$

A3. $x \oplus \zeta = \zeta \oplus x = x \iff x\zeta = \zeta x = x$ (holds for $\zeta = 1$)

A4. $x \oplus \eta = \eta \oplus x = 1 \iff x\eta = \eta x = 1$ (holds for $\eta = x^{-1}$)

A5. $r \odot (x \oplus y) = (r \odot x) \oplus (r \odot y) \iff (xy)^r = x^r y^r$

A6. $(r + s) \odot x = (r \odot x) \oplus (s \odot x) \iff x^{r+s} = x^r x^s$

A7. $(rs) \odot x = r \odot (s \odot x) \iff x^{rs} = (x^s)^r$

A8. $1 \odot x = x \iff x^1 = x$

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Examples.

- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

$C(\mathbb{R})$ is a subspace of $F(\mathbb{R})$.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_k x^k$
- $\mathcal{P}_n$: polynomials of degree less than n

$\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

Subspaces of vector spaces

Counterexamples.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$
- P_n^*: polynomials of degree n $(n > 0)$

P_n^* is not a subspace of $\mathcal{P}$.

$-x^n + (x^n + 1) = 1 \notin P_n^* \implies P_n^*$ is not a vector space (addition is not well defined).

- $\mathbb{R}$ with the standard linear operations
- $\mathbb{R}_+$ with the operations $\oplus$ and $\odot$

$\mathbb{R}_+$ is not a subspace of $\mathbb{R}$ since the linear operations do not agree.

If S is a subset of a vector space V then S inherits from V addition and scalar multiplication. However S need not be closed under these operations.

Proposition A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

Proof: "only if" is obvious.
"if": properties like associative, commutative, or distributive law hold for S because they hold for V. We only need to verify properties A3 and A4. Take any $\mathbf{x} \in S$ (note that S is nonempty). Then $\mathbf{0} = 0\mathbf{x} \in S$. Also, $-\mathbf{x} = (-1)\mathbf{x} \in S$. Thus $\mathbf{0}$ and $-\mathbf{x}$ in S are the same as in V.

Example. $V = \mathbb{R}^2$.

- The line $x - y = 0$ is a subspace of $\mathbb{R}^2$.

The line consists of all vectors of the form (t, t), $t \in \mathbb{R}$.

$(t, t) + (s, s) = (t + s, t + s) \implies$ closed under addition
$r(t, t) = (rt, rt) \implies$ closed under scaling

- The parabola $y = x^2$ is not a subspace of $\mathbb{R}^2$.

It is enough to find one explicit counterexample.

Counterexample 1: $(1, 1) + (-1, 1) = (0, 2)$.

$(1, 1)$ and $(-1, 1)$ lie on the parabola while $(0, 2)$ does not
$\implies$ not closed under addition

Counterexample 2: $2(1, 1) = (2, 2)$.

$(1, 1)$ lies on the parabola while $(2, 2)$ does not
$\implies$ not closed under scaling

Example. $V = \mathbb{R}^3$.

- The plane $z = 0$ is a subspace of $\mathbb{R}^3$.
- The plane $z = 1$ is not a subspace of $\mathbb{R}^3$.
- The line $t(1, 1, 0)$, $t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ and a subspace of the plane $z = 0$.
- The line $(1, 1, 1) + t(1, -1, 0)$, $t \in \mathbb{R}$ is not a subspace of $\mathbb{R}^3$ as it lies in the plane $x + y + z = 3$, which does not contain $\mathbf{0}$.
- In general, a straight line or a plane in $\mathbb{R}^3$ is a subspace if and only if it passes through the origin.