

MATH 304  
Linear Algebra

**Lecture 19:**  
**Review for Test 1.**

## Topics for Test 1

*Part I: Elementary linear algebra (Leon 1.1–1.5, 2.1–2.2)*

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.
- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.
- Matrix algebra. Inverse matrix.
- Determinants: explicit formulas for  $2 \times 2$  and  $3 \times 3$  matrices, row and column expansions, elementary row and column operations.

## Topics for Test 1

*Part II: Abstract linear algebra (Leon 3.1–3.4, 3.6)*

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set.
- Linear independence.
- Basis and dimension.
- Rank and nullity of a matrix.

## Sample problems for Test 1

**Problem 1** Find a quadratic polynomial  $p(x)$  such that  $p(1) = 1$ ,  $p(2) = 3$ , and  $p(3) = 7$ .

**Problem 2** Let  $A$  be a square matrix such that  $A^3 = O$ .

- (i) Prove that the matrix  $A$  is not invertible.
- (ii) Prove that the matrix  $A + I$  is invertible.

**Problem 3** Let  $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$ .

- (i) Evaluate the determinant of the matrix  $A$ .
- (ii) Find the inverse matrix  $A^{-1}$ .

## Sample problems for Test 1

**Problem 4** Determine which of the following subsets of  $\mathbb{R}^3$  are subspaces. Briefly explain.

- (i) The set  $S_1$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $xyz = 0$ .
- (ii) The set  $S_2$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $x + y + z = 0$ .
- (iii) The set  $S_3$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $y^2 + z^2 = 0$ .
- (iv) The set  $S_4$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $y^2 - z^2 = 0$ .

**Problem 5** Let  $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$ .

- (i) Find the rank and the nullity of the matrix  $B$ .
- (ii) Find a basis for the row space of  $B$ , then extend this basis to a basis for  $\mathbb{R}^4$ .
- (iii) Find a basis for the nullspace of  $B$ .

## Sample problems for Test 1

**Problem 6** Show that the functions  $f_1(x) = x$ ,  $f_2(x) = xe^x$ , and  $f_3(x) = e^{-x}$  are linearly independent in the vector space  $C^\infty(\mathbb{R})$ .

**Problem 7** Let  $V$  be a finite-dimensional vector space and  $V_0$  be a proper subspace of  $V$  (where proper means that  $V_0 \neq V$ ). Prove that  $\dim V_0 < \dim V$ .

**Problem 1.** Find a quadratic polynomial  $p(x)$  such that  $p(1) = 1$ ,  $p(2) = 3$ , and  $p(3) = 7$ .

Let  $p(x) = a + bx + cx^2$ . Then  $p(1) = a + b + c$ ,  $p(2) = a + 2b + 4c$ , and  $p(3) = a + 3b + 9c$ .

The coefficients  $a$ ,  $b$ , and  $c$  have to be chosen so that

$$\begin{cases} a + b + c = 1, \\ a + 2b + 4c = 3, \\ a + 3b + 9c = 7. \end{cases}$$

We solve this system of linear equations using elementary operations:

$$\begin{cases} a + b + c = 1 \\ a + 2b + 4c = 3 \\ a + 3b + 9c = 7 \end{cases} \iff \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ a + 3b + 9c = 7 \end{cases}$$

$$\Leftrightarrow \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ a + 3b + 9c = 7 \end{cases} \Leftrightarrow \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ 2b + 8c = 6 \end{cases}$$

$$\Leftrightarrow \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ b + 4c = 3 \end{cases} \Leftrightarrow \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ c = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a + b + c = 1 \\ b = -1 \\ c = 1 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = -1 \\ c = 1 \end{cases}$$

Thus the desired polynomial is  $p(x) = x^2 - x + 1$ .



**Problem 2.** Let  $A$  be a square matrix such that  $A^3 = O$ .

(i) Prove that the matrix  $A$  is not invertible.

The proof is by contradiction. Assume that  $A$  is invertible. Since any product of invertible matrices is also invertible, the matrix  $A^3 = AAA$  should be invertible as well. However  $A^3 = O$  is singular.

**Problem 2.** Let  $A$  be a square matrix such that  $A^3 = O$ .

(ii) Prove that the matrix  $A + I$  is invertible.

It is enough to show that the equation  $(A + I)\mathbf{x} = \mathbf{0}$  (where  $\mathbf{x}$  and  $\mathbf{0}$  are column vectors) has a unique solution  $\mathbf{x} = \mathbf{0}$ .

Indeed,  $(A + I)\mathbf{x} = \mathbf{0} \implies A\mathbf{x} + I\mathbf{x} = \mathbf{0} \implies A\mathbf{x} = -\mathbf{x}$ .

Then  $A^2\mathbf{x} = A(A\mathbf{x}) = A(-\mathbf{x}) = -A\mathbf{x} = -(-\mathbf{x}) = \mathbf{x}$ .

Further,  $A^3\mathbf{x} = A(A^2\mathbf{x}) = A\mathbf{x} = -\mathbf{x}$ . On the other hand,  $A^3\mathbf{x} = O\mathbf{x} = \mathbf{0}$ . Hence  $-\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$ .

Alternatively, we can use equalities

$$X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2) = (X^2 - XY + Y^2)(X + Y),$$

which hold whenever matrices  $X$  and  $Y$  commute:  $XY = YX$ .

In particular, they hold for  $X = A$  and  $Y = I$ . We obtain

$$(A + I)(A^2 - A + I) = (A^2 - A + I)(A + I) = A^3 + I^3 = I$$

so that  $(A + I)^{-1} = A^2 - A + I$ .

**Problem 3.** Let  $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$ .

(i) Evaluate the determinant of the matrix  $A$ .

Subtract the 4th row of  $A$  from the 3rd row:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 3rd row:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 3rd column:

$$(-1) \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix} = (-1) \left( \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} \right) = -1.$$

**Problem 3.** Let  $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$ .

(ii) Find the inverse matrix  $A^{-1}$ .

First we merge the matrix  $A$  with the identity matrix into one  $4 \times 8$  matrix

$$(A|I) = \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract 2 times the 1st row from the 2nd row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

Subtract 2 times the 1st row from the 3rd row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

Subtract 2 times the 1st row from the 4th row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array} \right)$$

Subtract 2 times the 4th row from the 2nd row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array} \right)$$

Subtract the 4th row from the 3rd row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array} \right)$$

Add 4 times the 2nd row to the 4th row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array} \right)$$

Add 32 times the 3rd row to the 4th row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right)$$

Add 10 times the 3rd row to the 2nd row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right)$$

Add the 4th row to the 1st row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right)$$



Add 4 times the 3rd row to the 1st row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right)$$

Subtract 2 times the 2nd row from the 1st row:

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right)$$

Multiply the 2nd, the 3rd, and the 4th rows by  $-1$ :

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array} \right)$$

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array} \right) = (I | A^{-1})$$

Finally the left part of our  $4 \times 8$  matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of  $A$ . Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}.$$

**Problem 3.** Let  $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$ .

(i) Evaluate the determinant of the matrix  $A$ .

*Alternative solution:* We have transformed  $A$  into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by  $-1$ .

It follows that  $\det I = (-1)^3 \det A$ .

$$\implies \det A = -\det I = -1.$$

**Problem 4.** Determine which of the following subsets of  $\mathbb{R}^3$  are subspaces. Briefly explain.

A subset of  $\mathbb{R}^3$  is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(i) The set  $S_1$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $xyz = 0$ .

$(0, 0, 0) \in S_1 \implies S_1$  is not empty.

$xyz = 0 \implies (rx)(ry)(rz) = r^3xyz = 0$ .

That is,  $\mathbf{v} = (x, y, z) \in S_1 \implies r\mathbf{v} = (rx, ry, rz) \in S_1$ .

Hence  $S_1$  is closed under scalar multiplication.

However  $S_1$  is not closed under addition.

Counterexample:  $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$ .

**Problem 4.** Determine which of the following subsets of  $\mathbb{R}^3$  are subspaces. Briefly explain.

A subset of  $\mathbb{R}^3$  is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(ii) The set  $S_2$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $x + y + z = 0$ .

$(0, 0, 0) \in S_2 \implies S_2$  is not empty.

$x + y + z = 0 \implies rx + ry + rz = r(x + y + z) = 0$ .

Hence  $S_2$  is closed under scalar multiplication.

$x + y + z = x' + y' + z' = 0 \implies$

$(x + x') + (y + y') + (z + z') = (x + y + z) + (x' + y' + z') = 0$ .

That is,  $\mathbf{v} = (x, y, z)$ ,  $\mathbf{v}' = (x', y', z') \in S_2$

$\implies \mathbf{v} + \mathbf{v}' = (x + x', y + y', z + z') \in S_2$ .

Hence  $S_2$  is closed under addition.

**(iii)** The set  $S_3$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $y^2 + z^2 = 0$ .

$$y^2 + z^2 = 0 \iff y = z = 0.$$

$S_3$  is a nonempty set closed under addition and scalar multiplication.

**(iv)** The set  $S_4$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $y^2 - z^2 = 0$ .

$S_4$  is a nonempty set closed under scalar multiplication. However  $S_4$  is not closed under addition.

Counterexample:  $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$ .

**Problem 5.** Let  $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$ .

(i) Find the rank and the nullity of the matrix  $B$ .

The rank (= dimension of the row space) and the nullity (= dimension of the nullspace) of a matrix are preserved under elementary row operations. We apply such operations to convert the matrix  $B$  into row echelon form.

Interchange the 1st row with the 2nd row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

Add 3 times the 1st row to the 3rd row, then subtract 2 times the 1st row from the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Multiply the 2nd row by  $-1$ :

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Add the 4th row to the 3rd row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$



Add 3 times the 2nd row to the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix}$$

Add 16 times the 3rd row to the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now that the matrix is in row echelon form, its rank equals the number of nonzero rows, which is 3. Since

$(\text{rank of } B) + (\text{nullity of } B) = (\text{the number of columns of } B) = 4,$   
it follows that the nullity of  $B$  equals 1.

**Problem 5.** Let  $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$ .

(ii) Find a basis for the row space of  $B$ , then extend this basis to a basis for  $\mathbb{R}^4$ .

The row space of a matrix is invariant under elementary row operations. Therefore the row space of the matrix  $B$  is the same as the row space of its row echelon form:

$$\begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The nonzero rows of the latter matrix are linearly independent so that they form a basis for its row space:

$$\mathbf{v}_1 = (1, 1, 2, -1), \quad \mathbf{v}_2 = (0, 1, -4, -1), \quad \mathbf{v}_3 = (0, 0, 1, 0).$$

To extend the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  to a basis for  $\mathbb{R}^4$ , we need a vector  $\mathbf{v}_4 \in \mathbb{R}^4$  that is not a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ .

It is known that at least one of the vectors  $\mathbf{e}_1 = (1, 0, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0, 0)$ ,  $\mathbf{e}_3 = (0, 0, 1, 0)$ , and  $\mathbf{e}_4 = (0, 0, 0, 1)$  can be chosen as  $\mathbf{v}_4$ .

In particular, the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_4$  form a basis for  $\mathbb{R}^4$ .

This follows from the fact that the  $4 \times 4$  matrix whose rows are these vectors is not singular:

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

**Problem 5.** Let  $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$ .

(iii) Find a basis for the nullspace of  $B$ .

The nullspace of  $B$  is the solution set of the system of linear homogeneous equations with  $B$  as the coefficient matrix. To solve the system, we convert  $B$  to reduced row echelon form:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\implies x_1 = x_2 - x_4 = x_3 = 0$$

General solution:  $(x_1, x_2, x_3, x_4) = (0, t, 0, t) = t(0, 1, 0, 1)$ .

Thus the vector  $(0, 1, 0, 1)$  forms a basis for the nullspace of  $B$ .

**Problem 6.** Show that the functions  $f_1(x) = x$ ,  $f_2(x) = xe^x$ , and  $f_3(x) = e^{-x}$  are linearly independent in the vector space  $C^\infty(\mathbb{R})$ .

The functions  $f_1, f_2, f_3$  are linearly independent whenever the Wronskian  $W[f_1, f_2, f_3]$  is not identically zero.

$$\begin{aligned} W[f_1, f_2, f_3](x) &= \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} = \begin{vmatrix} x & xe^x & e^{-x} \\ 1 & e^x + xe^x & -e^{-x} \\ 0 & 2e^x + xe^x & e^{-x} \end{vmatrix} \\ &= e^{-x} \begin{vmatrix} x & xe^x & 1 \\ 1 & e^x + xe^x & -1 \\ 0 & 2e^x + xe^x & 1 \end{vmatrix} = \begin{vmatrix} x & x & 1 \\ 1 & 1+x & -1 \\ 0 & 2+x & 1 \end{vmatrix} \\ &= x \begin{vmatrix} 1+x & -1 \\ 2+x & 1 \end{vmatrix} - \begin{vmatrix} x & 1 \\ 2+x & 1 \end{vmatrix} = x(2x+3) + 2 = 2x^2 + 3x + 2. \end{aligned}$$

The polynomial  $2x^2 + 3x + 2$  is never zero.

**Problem 6.** Show that the functions  $f_1(x) = x$ ,  $f_2(x) = xe^x$ , and  $f_3(x) = e^{-x}$  are linearly independent in the vector space  $C^\infty(\mathbb{R})$ .

*Alternative solution:* Suppose that  $af_1(x) + bf_2(x) + cf_3(x) = 0$  for all  $x \in \mathbb{R}$ , where  $a, b, c$  are constants. We have to show that  $a = b = c = 0$ .

Let us differentiate this identity:

$$ax + bxe^x + ce^{-x} = 0,$$

$$a + be^x + bxe^x - ce^{-x} = 0,$$

$$2be^x + bxe^x + ce^{-x} = 0,$$

$$3be^x + bxe^x - ce^{-x} = 0,$$

$$4be^x + bxe^x + ce^{-x} = 0.$$

(the 5th identity) – (the 3rd identity):  $2be^x = 0 \implies b = 0$ .

Substitute  $b = 0$  in the 3rd identity:  $ce^{-x} = 0 \implies c = 0$ .

Substitute  $b = c = 0$  in the 2nd identity:  $a = 0$ .

**Problem 6.** Show that the functions  $f_1(x) = x$ ,  $f_2(x) = xe^x$ , and  $f_3(x) = e^{-x}$  are linearly independent in the vector space  $C^\infty(\mathbb{R})$ .

*Alternative solution:* Suppose that  $ax + bxe^x + ce^{-x} = 0$  for all  $x \in \mathbb{R}$ , where  $a, b, c$  are constants. We have to show that  $a = b = c = 0$ .

For any  $x \neq 0$  divide both sides of the identity by  $xe^x$ :

$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$

The left-hand side approaches  $b$  as  $x \rightarrow +\infty$ .  $\implies b = 0$

Now  $ax + ce^{-x} = 0$  for all  $x \in \mathbb{R}$ . For any  $x \neq 0$  divide both sides of the identity by  $x$ :

$$a + cx^{-1}e^{-x} = 0.$$

The left-hand side approaches  $a$  as  $x \rightarrow +\infty$ .  $\implies a = 0$

Now  $ce^{-x} = 0 \implies c = 0$ .

**Problem 7.** Let  $V$  be a finite-dimensional vector space and  $V_0$  be a proper subspace of  $V$  (where proper means that  $V_0 \neq V$ ). Prove that  $\dim V_0 < \dim V$ .

Any vector space has a basis. Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be a basis for  $V_0$ .

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly independent in  $V$  since they are linearly independent in  $V_0$ . Therefore we can extend this collection of vectors to a basis for  $V$  by adding some vectors  $\mathbf{w}_1, \dots, \mathbf{w}_m$ . As  $V_0 \neq V$ , we do need to add some vectors, i.e.,  $m \geq 1$ .

Thus  $\dim V_0 = k$  and  $\dim V = k + m > k$ .