

MATH 304

Linear Algebra

Lecture 20:

Basis and coordinates.

Change of basis.

Basis and dimension

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Theorem Any vector space V has a basis. If V has a finite basis, then all bases for V are finite and have the same number of elements (called the *dimension* of V).

Example. Vectors $\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0), \dots, \mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$ form a basis for \mathbb{R}^n (called *standard*) since

$$(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients x_1, x_2, \dots, x_n are called the **coordinates** of \mathbf{v} with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

The mapping

$$\text{vector } \mathbf{v} \mapsto \text{its coordinates } (x_1, x_2, \dots, x_n)$$

is a one-to-one correspondence between V and \mathbb{R}^n . This correspondence respects linear operations in V and in \mathbb{R}^n .

Examples. • Coordinates of a vector

$\mathbf{v} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ relative to the standard basis $\mathbf{e}_1 = (1, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, \dots, 0, 0), \dots$, $\mathbf{e}_n = (0, 0, \dots, 0, 1)$ are (x_1, x_2, \dots, x_n) .

• Coordinates of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$

relative to the basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are (a, c, b, d) .

• Coordinates of a polynomial

$p(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} \in \mathcal{P}_n$ relative to the basis $1, x, x^2, \dots, x^{n-1}$ are $(a_0, a_1, \dots, a_{n-1})$.

Weird vector space

Consider the set $V = \mathbb{R}_+$ of positive numbers with a nonstandard addition and scalar multiplication:

$$\boxed{x \oplus y = xy} \quad \text{for any } x, y \in \mathbb{R}_+.$$

$$\boxed{r \odot x = x^r} \quad \text{for any } x \in \mathbb{R}_+ \text{ and } r \in \mathbb{R}.$$

This is an example of a vector space.

The zero vector in V is the number 1. To build a basis for V , we can begin with any number $v \in V$ different from 1. Let's take $v = 2$. The span $\text{Span}(2)$ consists of all numbers of the form $r \odot 2 = 2^r$, $r \in \mathbb{R}$. It is the entire space V . Hence $\{2\}$ is a basis for V so that $\dim V = 1$.

The coordinate mapping $f : V \rightarrow \mathbb{R}$ associated to this basis is given by $f(2^r) = r$ for all $r \in \mathbb{R}$. Equivalently, $f(x) = \log_2 x$, $x \in V$. Notice that $\log_2(x \oplus y) = \log_2 x + \log_2 y$ and $\log_2(r \odot x) = r \log_2 x$.

Vectors $\mathbf{u}_1=(3, 1)$ and $\mathbf{u}_2=(2, 1)$ form a basis for \mathbb{R}^2 .

Problem 1. Find coordinates of the vector $\mathbf{v} = (7, 4)$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2$.

The desired coordinates x, y satisfy

$$\mathbf{v} = x\mathbf{u}_1 + y\mathbf{u}_2 \iff \begin{cases} 3x + 2y = 7 \\ x + y = 4 \end{cases} \iff \begin{cases} x = -1 \\ y = 5 \end{cases}$$

Problem 2. Find the vector \mathbf{w} whose coordinates with respect to the basis $\mathbf{u}_1, \mathbf{u}_2$ are $(7, 4)$.

$$\mathbf{w} = 7\mathbf{u}_1 + 4\mathbf{u}_2 = 7(3, 1) + 4(2, 1) = (29, 11)$$

Change of coordinates

Given a vector $\mathbf{v} \in \mathbb{R}^2$, let (x, y) be its standard coordinates, i.e., coordinates with respect to the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, and let (x', y') be its coordinates with respect to the basis $\mathbf{u}_1 = (3, 1)$, $\mathbf{u}_2 = (2, 1)$.

Problem. Find a relation between (x, y) and (x', y') .

By definition, $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{u}_1 + y'\mathbf{u}_2$.

In standard coordinates,

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= x' \begin{pmatrix} 3 \\ 1 \end{pmatrix} + y' \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ \implies \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

Change of coordinates in \mathbb{R}^n

The usual (standard) coordinates of a vector

$\mathbf{v} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ are coordinates relative to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for \mathbb{R}^n and $(x'_1, x'_2, \dots, x'_n)$ be the coordinates of the same vector \mathbf{v} with respect to this basis. Then

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix},$$

where the matrix $U = (u_{ij})$ does not depend on the vector \mathbf{v} . Namely, columns of U are coordinates of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ with respect to the standard basis. U is called the **transition matrix** from the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. The inverse matrix U^{-1} is called the **transition matrix** from $\mathbf{e}_1, \dots, \mathbf{e}_n$ to $\mathbf{u}_1, \dots, \mathbf{u}_n$.

Problem. Find coordinates of the vector $\mathbf{x} = (1, 2, 3)$ with respect to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

The nonstandard coordinates (x', y', z') of \mathbf{x} satisfy

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = U \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

where U is the transition matrix from the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

The transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$U_0 = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \left(\begin{array}{c|c|c} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right).$$

The transition matrix from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is the inverse matrix: $U = U_0^{-1}$.

The inverse matrix can be computed using row reduction.

$$(U_0 | I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array} \right) = (I | U_0^{-1})$$

Thus

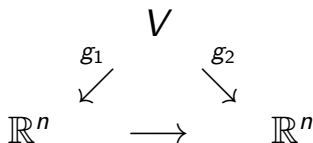
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Change of coordinates: general case

Let V be a vector space of dimension n .

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for V and $g_2 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.



The composition $g_2 \circ g_1^{-1}$ is a transformation of \mathbb{R}^n .

It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

Problem. Find the transition matrix from the basis $p_1(x) = 1$, $p_2(x) = x + 1$, $p_3(x) = (x + 1)^2$ to the basis $q_1(x) = 1$, $q_2(x) = x$, $q_3(x) = x^2$ for the vector space \mathcal{P}_3 .

We have to find coordinates of the polynomials p_1, p_2, p_3 with respect to the basis q_1, q_2, q_3 :

$$p_1(x) = 1 = q_1(x),$$

$$p_2(x) = x + 1 = q_1(x) + q_2(x),$$

$$p_3(x) = (x+1)^2 = x^2 + 2x + 1 = q_1(x) + 2q_2(x) + q_3(x).$$

Hence the transition matrix is
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the polynomial identity

$$a_1 + a_2(x + 1) + a_3(x + 1)^2 = b_1 + b_2x + b_3x^2$$

is equivalent to the relation

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Problem. Find the transition matrix from the basis $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (1, 2, 1)$ to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

It is convenient to make a two-step transition: first from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and then from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Let U_1 be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and U_2 be the transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$U_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \implies$ coordinates \mathbf{x}

Basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \implies$ coordinates $U_1\mathbf{x}$

Basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \implies$ coordinates $U_2^{-1}(U_1\mathbf{x}) = (U_2^{-1}U_1)\mathbf{x}$

Thus the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is $U_2^{-1}U_1$.

$$\begin{aligned} U_2^{-1}U_1 &= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}. \end{aligned}$$