MATH 304

Linear Algebra

Lecture 26:
Eigenvalues and eigenvectors (continued).
Basis of eigenvectors.
Diagonalization.

Eigenvalues and eigenvectors of an operator

Definition. Let V be a vector space and $L : V \to V$ be a linear operator. A number λ is called an **eigenvalue** of the operator L if $\boxed{L(\mathbf{v}) = \lambda\mathbf{v}}$ for a nonzero vector $\mathbf{v} \in V$. The vector $\mathbf{v}$ is called an **eigenvector** of L associated with the eigenvalue λ.

The set V_λ of all eigenvectors of L associated with the eigenvalue λ along with the zero vector is a subspace of V. It is called the **eigenspace** of L corresponding to the eigenvalue λ.

Example. $V = C^\infty(\mathbb{R})$, $D : V \to V$, $Df = f'$.

A function $f \in C^\infty(\mathbb{R})$ is an eigenfunction of the operator D belonging to an eigenvalue λ if $f'(x) = \lambda f(x)$ for all $x \in \mathbb{R}$.

It follows that $f(x) = ce^{\lambda x}$, where c is a nonzero constant.

Thus each $\lambda \in \mathbb{R}$ is an eigenvalue of D.
The corresponding eigenspace is spanned by $e^{\lambda x}$.

Example. $V = C^\infty(\mathbb{R})$, $L : V \to V$, $Lf = f''$.

$Lf = \lambda f \iff f''(x) - \lambda f(x) = 0$ for all $x \in \mathbb{R}$.

It follows that each $\lambda \in \mathbb{R}$ is an eigenvalue of L and the corresponding eigenspace V_λ is two-dimensional. Note that $L = D^2$, hence $Df = \mu f \implies Lf = \mu^2 f$.

If $\lambda > 0$ then $V_\lambda = \mathrm{Span}(e^{\mu x}, e^{-\mu x})$, where $\mu = \sqrt{\lambda}$.

If $\lambda < 0$ then $V_\lambda = \mathrm{Span}\big(\sin(\mu x), \cos(\mu x)\big)$, where $\mu = \sqrt{-\lambda}$.

If $\lambda = 0$ then $V_\lambda = \mathrm{Span}(1, x)$.

Let V be a vector space and $L : V \to V$ be a linear operator.

Proposition 1 If $\mathbf{v} \in V$ is an eigenvector of the operator L then the associated eigenvalue is unique.

Proof: Suppose that $L(\mathbf{v}) = \lambda_1 \mathbf{v}$ and $L(\mathbf{v}) = \lambda_2 \mathbf{v}$. Then
$$\lambda_1 \mathbf{v} = \lambda_2 \mathbf{v} \implies (\lambda_1 - \lambda_2)\mathbf{v} = \mathbf{0} \implies \lambda_1 - \lambda_2 = 0 \implies \lambda_1 = \lambda_2.$$

Proposition 2 Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of L associated with different eigenvalues λ_1 and λ_2. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Proof: For any scalar $t \neq 0$ the vector $t\mathbf{v}_1$ is also an eigenvector of L associated with the eigenvalue λ_1. Since $\lambda_2 \neq \lambda_1$, it follows that $\mathbf{v}_2 \neq t\mathbf{v}_1$. That is, $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$. Similarly, $\mathbf{v}_1$ is not a scalar multiple of $\mathbf{v}_2$.

Let $L : V \to V$ be a linear operator.

Proposition 3 If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of L associated with distinct eigenvalues λ_1, λ_2, and λ_3, then they are linearly independent.

Proof: Suppose that $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3 = \mathbf{0}$ for some $t_1, t_2, t_3 \in \mathbb{R}$. Then

$$L(t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3) = \mathbf{0},$$
$$t_1L(\mathbf{v}_1) + t_2L(\mathbf{v}_2) + t_3L(\mathbf{v}_3) = \mathbf{0},$$
$$t_1\lambda_1\mathbf{v}_1 + t_2\lambda_2\mathbf{v}_2 + t_3\lambda_3\mathbf{v}_3 = \mathbf{0}.$$

It follows that

$$t_1\lambda_1\mathbf{v}_1 + t_2\lambda_2\mathbf{v}_2 + t_3\lambda_3\mathbf{v}_3 - \lambda_3(t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3) = \mathbf{0}$$
$$\implies t_1(\lambda_1 - \lambda_3)\mathbf{v}_1 + t_2(\lambda_2 - \lambda_3)\mathbf{v}_2 = \mathbf{0}.$$

By the above, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.
Hence $t_1(\lambda_1 - \lambda_3) = t_2(\lambda_2 - \lambda_3) = 0 \implies t_1 = t_2 = 0$
Then $t_3 = 0$ as well.

Theorem If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of a linear operator L associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Corollary If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct real numbers, then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}$ are linearly independent.

Proof: Consider a linear operator $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by $Df = f'$. Then $e^{\lambda_1 x}, \ldots, e^{\lambda_k x}$ are eigenfunctions of D associated with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. By the theorem, the eigenfunctions are linearly independent.

Basis of eigenvectors

Let V be a finite-dimensional vector space and $L : V \to V$ be a linear operator. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and A be the matrix of the operator L with respect to this basis.

Theorem The matrix A is diagonal if and only if vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are eigenvectors of L.

If this is the case, then the diagonal entries of the matrix A are the corresponding eigenvalues of L.

$$L(\mathbf{v}_i) = \lambda_i \mathbf{v}_i \iff A = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix}$$

How to find a basis of eigenvectors

Theorem If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of a linear operator L associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Corollary 1 Suppose $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all eigenvalues of a linear operator $L : V \to V$. For any $1 \leq i \leq k$, let S_i be a basis for the eigenspace associated to the eigenvalue λ_i. Then these bases are disjoint and the union $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent set.

Moreover, if the vector space V admits a basis consisting of eigenvectors of L, then S is such a basis.

Corollary 2 Let A be an $n \times n$ matrix such that the characteristic equation $\det(A - \lambda I) = 0$ has n distinct roots. Then **(i)** there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A; **(ii)** all eigenspaces of A are one-dimensional.

Diagonalization

Theorem 1 Let L be a linear operator on a finite-dimensional vector space V. Then the following conditions are equivalent:

- the matrix of L with respect to some basis is diagonal;
- there exists a basis for V formed by eigenvectors of L.

The operator L is **diagonalizable** if it satisfies these conditions.

Theorem 2 Let A be an $n \times n$ matrix. Then the following conditions are equivalent:

- A is the matrix of a diagonalizable operator;
- A is similar to a diagonal matrix, i.e., it is represented as $A = UBU^{-1}$, where the matrix B is diagonal;
- there exists a basis for $\mathbb{R}^n$ formed by eigenvectors of A.

The matrix A is **diagonalizable** if it satisfies these conditions.

Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 1 and 3.
- The eigenspace of A associated with the eigenvalue 1 is the line spanned by $\mathbf{v}_1 = (-1, 1)$.
- The eigenspace of A associated with the eigenvalue 3 is the line spanned by $\mathbf{v}_2 = (1, 1)$.
- Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$.

Thus the matrix A is diagonalizable. Namely, $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Notice that U is the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2$ to the standard basis.

Example. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 0 and 2.
- The eigenspace for 0 is one-dimensional; it has a basis $S_1 = \{\mathbf{v}_1\}$, where $\mathbf{v}_1 = (-1, 1, 0)$.
- The eigenspace for 2 is two-dimensional; it has a basis $S_2 = \{\mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (-1, 0, 1)$.
- The union $S_1 \cup S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, hence it is a basis for $\mathbb{R}^3$.

Thus the matrix A is diagonalizable. Namely, $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$