MATH 304

Linear Algebra

Lecture 32:
Review for Test 2.

Topics for Test 2

Coordinates and linear transformations (Leon 3.5, 4.1–4.3)

- Coordinates relative to a basis
- Change of basis, transition matrix
- Linear transformations
- Matrix of a linear transformation
- Change of basis for a linear operator

Eigenvalues and eigenvectors (Leon 6.1, 6.3)

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Diagonalization

Orthogonality (Leon 5.1–5.3, 5.5–5.6)

- Orthogonal complement
- Orthogonal projection
- Least squares problems
- The Gram-Schmidt orthogonalization process

Sample problems for Test 2

Problem 1. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(\mathbf{u}) = (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_2$, where $\mathbf{v}_1 = (1, 2, -1)$ and $\mathbf{v}_2 = (1, 2, 3)$.

(i) Find a matrix M such that $L(\mathbf{u}) = M\mathbf{u}$ for any column vector $\mathbf{u} \in \mathbb{R}^3$.

(ii) Find all eigenvalues and eigenvectors of L.

Problem 2. Let V be a subspace of $F(\mathbb{R})$ spanned by functions e^x and e^{-x}. Let L be a linear operator on V such that $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the matrix of L relative to the basis e^x, e^{-x}. Find the matrix of L relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.
(ii) For each eigenvalue of A, find an associated eigenvector.
(iii) Is the matrix A diagonalizable? Explain.
(iv) Find all eigenvalues of the matrix A^2.

Problem 4. Find a linear polynomial which is the best least squares fit to the following data:

x	-2	-1	0	1	2
$f(x)$	-3	-2	1	2	5

Problem 5. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.
(i) Find an orthonormal basis for V.
(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.

Problem 6. Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space V to a vector space W. Show that

$$\dim \mathrm{Range}(L) + \dim \ker(L) = \dim V.$$

Problem 7. Prove that every subspace of $\mathbb{R}^n$ is the solution set for some system of linear homogeneous equations in n variables.

Problem 1. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(\mathbf{u}) = (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_2$, where $\mathbf{v}_1 = (1, 2, -1)$ and $\mathbf{v}_2 = (1, 2, 3)$.

(i) Find a matrix M such that $L(\mathbf{u}) = M\mathbf{u}$ for any column vector $\mathbf{u} \in \mathbb{R}^3$.

(ii) Find all eigenvalues and eigenvectors of L.

M is the matrix of the operator L relative to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Hence consecutive columns of M are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$. We obtain that $L(\mathbf{e}_1) = \mathbf{v}_2$, $L(\mathbf{e}_2) = 2\mathbf{v}_2$, $L(\mathbf{e}_3) = -\mathbf{v}_2$.

Hence $M = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (1, 2, -1).$

Since $L(\mathbf{u})$ is always parallel to $\mathbf{v}_2$, it follows that L has two eigenspaces: the kernel, which is $\mathbf{v}_1^{\perp}$, and the line spanned by $\mathbf{v}_2$. We have $L(\mathbf{v}_2) = 2\mathbf{v}_2$ so that the eigenvalues are 0 and 2.

Problem 2. Let V be a subspace of $F(\mathbb{R})$ spanned by functions e^x and e^{-x}. Let L be a linear operator on V such that $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the matrix of L relative to the basis e^x, e^{-x}. Find the matrix of L relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Let A denote the matrix of the operator L relative to the basis e^x, e^{-x} (which is given) and B denote the matrix of L relative to the basis $\cosh x$, $\sinh x$ (which is to be found). By definition of the functions $\cosh x$ and $\sinh x$, the transition matrix from $\cosh x$, $\sinh x$ to e^x, e^{-x} is $U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. It follows that $B = U^{-1}AU$. We obtain that

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}.$$

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.

The eigenvalues of A are roots of the characteristic equation $\det(A - \lambda I) = 0$. We obtain that

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^3 - 2(1-\lambda) - 2(1-\lambda) = (1-\lambda)\big((1-\lambda)^2 - 4\big)$$

$$= (1-\lambda)\big((1-\lambda) - 2\big)\big((1-\lambda) + 2\big) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix A has three eigenvalues: -1, 1, and 3.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(ii) For each eigenvalue of A, find an associated eigenvector.

An eigenvector $\mathbf{v} = (x, y, z)$ of the matrix A associated with an eigenvalue λ is a nonzero solution of the vector equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve the equation, we convert the matrix $A - \lambda I$ to reduced row echelon form.

First consider the case $\lambda = -1$. The row reduction yields

$$A + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A + I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)$ is an eigenvector of A associated with the eigenvalue -1.

Secondly, consider the case $\lambda = 1$. The row reduction yields

$$A - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, 0, 1)$ is an eigenvector of A associated with the eigenvalue 1.

Finally, consider the case $\lambda = 3$. The row reduction yields

$$A-3I = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A-3I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = t, \ y = t, \ z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1,1,1)$ is an eigenvector of A associated with the eigenvalue 3.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(iii) Is the matrix A diagonalizable? Explain.

The matrix A is diagonalizable, i.e., there exists a basis for $\mathbb{R}^3$ formed by its eigenvectors.

Namely, the vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix A belonging to distinct eigenvalues. Therefore these vectors are linearly independent. It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

Alternatively, the existence of a basis for $\mathbb{R}^3$ consisting of eigenvectors of A already follows from the fact that the matrix A has three distinct eigenvalues.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(iv) Find all eigenvalues of the matrix A^2.

Suppose that $\mathbf{v}$ is an eigenvector of the matrix A associated with an eigenvalue λ, that is, $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

Therefore $\mathbf{v}$ is also an eigenvector of the matrix A^2 and the associated eigenvalue is λ^2. We already know that the matrix A has eigenvalues -1, 1, and 3. It follows that A^2 has eigenvalues 1 and 9.

Since a 3×3 matrix can have up to 3 eigenvalues, we need an additional argument to show that 1 and 9 are the only eigenvalues of A^2. One reason is that the eigenvalue 1 has multiplicity 2.

Problem 4. Find a linear polynomial which is the best least squares fit to the following data:

x	-2	-1	0	1	2
$f(x)$	-3	-2	1	2	5

We are looking for a function $f(x) = c_1 + c_2 x$, where c_1, c_2 are unknown coefficients. The data of the problem give rise to an overdetermined system of linear equations in variables c_1 and c_2:

$$\begin{cases} c_1 - 2c_2 = -3, \\ c_1 - c_2 = -2, \\ c_1 = 1, \\ c_1 + c_2 = 2, \\ c_1 + 2c_2 = 5. \end{cases}$$

This system is inconsistent.

We can represent the system as a matrix equation $A\mathbf{c} = \mathbf{y}$, where

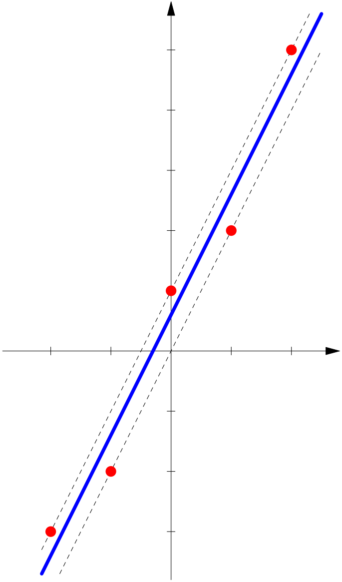
$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

The least squares solution $\mathbf{c}$ of the above system is a solution of the normal system $A^T A\mathbf{c} = A^T\mathbf{y}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\iff \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 20 \end{pmatrix} \iff \begin{cases} c_1 = 3/5 \\ c_2 = 2 \end{cases}$$

Thus the function $f(x) = \frac{3}{5} + 2x$ is the best least squares fit to the above data among linear polynomials.



Problem 5. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

(i) Find an orthonormal basis for V.

First we apply the Gram-Schmidt orthogonalization process to vectors $\mathbf{x}_1, \mathbf{x}_2$ and obtain an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2$ for the subspace V:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1, 1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1, 0, 3, 0) - \frac{4}{4}(1, 1, 1, 1) = (0, -1, 2, -1).$$

Then we normalize vectors $\mathbf{v}_1, \mathbf{v}_2$ to obtain an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2$ for V:

$$\|\mathbf{v}_1\| = 2 \implies \mathbf{w}_1 = \tfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \tfrac{1}{2}(1, 1, 1, 1)$$

$$\|\mathbf{v}_2\| = \sqrt{6} \implies \mathbf{w}_2 = \tfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \tfrac{1}{\sqrt{6}}(0, -1, 2, -1)$$

Problem 5. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.

(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.

Since the subspace V is spanned by vectors $(1,1,1,1)$ and $(1,0,3,0)$, it is the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

Then the orthogonal complement $V^\perp$ is the nullspace of A. To find the nullspace, we convert the matrix A to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

Hence a vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ belongs to $V^\perp$ if and only if

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \quad \begin{cases} x_1 + 3x_3 = 0 \\ x_2 - 2x_3 + x_4 = 0 \end{cases} \quad \iff \quad \begin{cases} x_1 = -3x_3 \\ x_2 = 2x_3 - x_4 \end{cases}$$

The general solution of the system is $(x_1, x_2, x_3, x_4) =$
$= (-3t, 2t - s, t, s) = t(-3, 2, 1, 0) + s(0, -1, 0, 1),$ where $t, s \in \mathbb{R}$.

It follows that $V^\perp$ is spanned by vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$.

The vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$ form a basis for the subspace $V^\perp$.

It remains to orthogonalize and normalize this basis:

$$\mathbf{v}_3 = \mathbf{x}_3 = (0, -1, 0, 1),$$

$$\mathbf{v}_4 = \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\,\mathbf{v}_3 = (-3, 2, 1, 0) - \frac{-2}{2}(0, -1, 0, 1)$$
$$= (-3, 1, 1, 1),$$

$$\|\mathbf{v}_3\| = \sqrt{2} \implies \mathbf{w}_3 = \tfrac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \tfrac{1}{\sqrt{2}}(0, -1, 0, 1),$$

$$\|\mathbf{v}_4\| = \sqrt{12} = 2\sqrt{3} \implies \mathbf{w}_4 = \tfrac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \tfrac{1}{2\sqrt{3}}(-3, 1, 1, 1).$$

Thus the vectors $\mathbf{w}_3 = \tfrac{1}{\sqrt{2}}(0, -1, 0, 1)$ and $\mathbf{w}_4 = \tfrac{1}{2\sqrt{3}}(-3, 1, 1, 1)$ form an orthonormal basis for $V^\perp$.

Problem 5. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.
(i) Find an orthonormal basis for V.
(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.

Alternative solution: First we extend the set $\mathbf{x}_1, \mathbf{x}_2$ to a basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ for $\mathbb{R}^4$. Then we orthogonalize and normalize the latter. This yields an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ for $\mathbb{R}^4$.

By construction, $\mathbf{w}_1, \mathbf{w}_2$ is an orthonormal basis for V.
It follows that $\mathbf{w}_3, \mathbf{w}_4$ is an orthonormal basis for $V^\perp$.

The set $\mathbf{x}_1 = (1,1,1,1)$, $\mathbf{x}_2 = (1,0,3,0)$ can be extended to a basis for $\mathbb{R}^4$ by adding two vectors from the standard basis.

For example, we can add vectors $\mathbf{e}_3 = (0,0,1,0)$ and $\mathbf{e}_4 = (0,0,0,1)$. To show that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{e}_3, \mathbf{e}_4$ is indeed a basis for $\mathbb{R}^4$, we check that the matrix whose rows are these vectors is nonsingular:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0.$$

To orthogonalize the basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{e}_3, \mathbf{e}_4$, we apply the Gram-Schmidt process:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1,1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1,0,3,0) - \tfrac{4}{4}(1,1,1,1) = (0,-1,2,-1),$$

$$\mathbf{v}_3 = \mathbf{e}_3 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (0,0,1,0) - \tfrac{1}{4}(1,1,1,1) -$$
$$- \tfrac{2}{6}(0,-1,2,-1) = \left(-\tfrac{1}{4}, \tfrac{1}{12}, \tfrac{1}{12}, \tfrac{1}{12}\right) = \tfrac{1}{12}(-3,1,1,1),$$

$$\mathbf{v}_4 = \mathbf{e}_4 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 = (0,0,0,1) -$$
$$- \tfrac{1}{4}(1,1,1,1) - \tfrac{-1}{6}(0,-1,2,-1) - \tfrac{1/12}{1/12} \cdot \tfrac{1}{12}(-3,1,1,1) =$$
$$= \left(0, -\tfrac{1}{2}, 0, \tfrac{1}{2}\right) = \tfrac{1}{2}(0,-1,0,1).$$

It remains to normalize vectors $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (0, -1, 2, -1)$, $\mathbf{v}_3 = \frac{1}{12}(-3, 1, 1, 1)$, $\mathbf{v}_4 = \frac{1}{2}(0, -1, 0, 1)$:

$$\|\mathbf{v}_1\| = 2 \implies \mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}(1, 1, 1, 1)$$

$$\|\mathbf{v}_2\| = \sqrt{6} \implies \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}(0, -1, 2, -1)$$

$$\|\mathbf{v}_3\| = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} \implies \mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$$

$$\|\mathbf{v}_4\| = \frac{1}{\sqrt{2}} \implies \mathbf{w}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$$

Thus $\mathbf{w}_1, \mathbf{w}_2$ is an orthonormal basis for V while $\mathbf{w}_3, \mathbf{w}_4$ is an orthonormal basis for $V^{\perp}$.

Thus for any vector $\mathbf{y} \in \mathbb{R}^4$ the orthogonal projection of $\mathbf{y}$ onto the subspace V is

$$\mathbf{p} = (\mathbf{y} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{y} \cdot \mathbf{w}_2)\mathbf{w}_2$$

and the orthogonal projection of $\mathbf{y}$ onto $V^\perp$ is

$$\mathbf{o} = (\mathbf{y} \cdot \mathbf{w}_3)\mathbf{w}_3 + (\mathbf{y} \cdot \mathbf{w}_4)\mathbf{w}_4.$$

Also, the distance from $\mathbf{y}$ to V is $\|\mathbf{y} - \mathbf{p}\| = \|\mathbf{o}\|$ and the distance from $\mathbf{y}$ to $V^\perp$ is $\|\mathbf{y} - \mathbf{o}\| = \|\mathbf{p}\|$.

Problem 6. Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space V to a vector space W. Show that $\dim \mathrm{Range}(L) + \dim \ker(L) = \dim V$.

The kernel $\ker(L)$ is a subspace of V. It is finite-dimensional since the vector space V is.

Take a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ for the subspace $\ker(L)$, then extend it to a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ for the entire space V.

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range of L.

Assuming the claim is proved, we obtain

$$\dim \mathrm{Range}(L) = m, \quad \dim \ker(L) = k, \quad \dim V = k + m.$$

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \ldots, L(\mathbf{u}_m)$ form a basis for the range of L.

Proof (spanning): Any vector $\mathbf{w} \in \mathrm{Range}(L)$ is represented as $\mathbf{w} = L(\mathbf{v})$, where $\mathbf{v} \in V$. Then

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k + \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \cdots + \beta_m \mathbf{u}_m$$

for some $\alpha_i, \beta_j \in \mathbb{R}$. It follows that

$$\mathbf{w} = L(\mathbf{v}) = \alpha_1 L(\mathbf{v}_1) + \cdots + \alpha_k L(\mathbf{v}_k) + \beta_1 L(\mathbf{u}_1) + \cdots + \beta_m L(\mathbf{u}_m)$$

$$= \beta_1 L(\mathbf{u}_1) + \cdots + \beta_m L(\mathbf{u}_m).$$

Note that $L(\mathbf{v}_i) = \mathbf{0}$ since $\mathbf{v}_i \in \ker(L)$.
Thus $\mathrm{Range}(L)$ is spanned by the vectors $L(\mathbf{u}_1), \ldots, L(\mathbf{u}_m)$.

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range of L.

Proof (linear independence): Suppose that

$$t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \cdots + t_m L(\mathbf{u}_m) = \mathbf{0}$$

for some $t_i \in \mathbb{R}$. Let $\mathbf{u} = t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_m\mathbf{u}_m$. Since

$$L(\mathbf{u}) = t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \cdots + t_m L(\mathbf{u}_m) = \mathbf{0},$$

the vector $\mathbf{u}$ belongs to the kernel of L. Therefore $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k$ for some $s_j \in \mathbb{R}$. It follows that

$$t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_m\mathbf{u}_m - s_1\mathbf{v}_1 - s_2\mathbf{v}_2 - \cdots - s_k\mathbf{v}_k = \mathbf{u} - \mathbf{u} = \mathbf{0}.$$

Linear independence of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m$ implies that $t_1 = \cdots = t_m = 0$ (as well as $s_1 = \cdots = s_k = 0$). Thus the vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ are linearly independent.

Problem 7. Prove that every subspace of $\mathbb{R}^n$ is the solution set for some system of linear homogeneous equations in n variables.

The proof is based on 3 observations:

(1) the solution set for a system of linear homogeneous equations is the nullspace of the coefficient matrix;
(2) for any matrix, $\{\text{nullspace}\} = \{\text{row space}\}^\perp$;
(3) any subspace of $\mathbb{R}^n$ can be represented as the row space of some matrix.

Now, given a subspace $V \subset \mathbb{R}^n$, let $W = V^\perp$. By the above, there exists a matrix A such that $W = \{\text{row space of } A\}$. Then $V = W^\perp = \{\text{nullspace of } A\}$.