MATH 304

Linear Algebra

Lecture 39:
Review for the final exam.

Topics for the final exam: Part I

Elementary linear algebra (Leon 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (Leon 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.
- Linear transformations.
- Matrix of a linear transformation.
- Change of basis for a linear operator.
- Similarity of matrices.

Topics for the final exam: Parts III–IV

Advanced linear algebra (Leon 5.1–5.7, 6.1–6.3)

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product).
- Inner products and norms.
- Orthogonal complement, orthogonal projection.
- Least squares problems.
- The Gram-Schmidt orthogonalization process.
- Orthogonal polynomials.

- Eigenvalues, eigenvectors, eigenspaces.
- Characteristic polynomial.
- Bases of eigenvectors, diagonalization.
- Matrix exponentials.
- Complex eigenvalues and eigenvectors.
- Orthogonal matrices.
- Rigid motions, rotations in space.

Bases of eigenvectors

Let A be an $n \times n$ matrix with real entries.

- A has n distinct real eigenvalues $\Longrightarrow$ a basis for $\mathbb{R}^n$ formed by eigenvectors of A
- A has complex eigenvalues $\Longrightarrow$ no basis for $\mathbb{R}^n$ formed by eigenvectors of A
- A has n distinct complex eigenvalues $\Longrightarrow$ a basis for $\mathbb{C}^n$ formed by eigenvectors of A
- A has multiple eigenvalues $\Longrightarrow$ further information is needed

- an orthonormal basis for $\mathbb{R}^n$ formed by eigenvectors of A $\Longleftrightarrow$ A is symmetric: $A^T = A$

Problem. For each of the following 2×2 matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^2$,
(b) a basis of eigenvectors for $\mathbb{C}^2$,
(c) an orthonormal basis of eigenvectors for $\mathbb{R}^2$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$ (a),(b),(c): yes

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ (a),(b),(c): no

Problem. For each of the following 2×2 matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^2$,

(b) a basis of eigenvectors for $\mathbb{C}^2$,

(c) an orthonormal basis of eigenvectors for $\mathbb{R}^2$.

$C = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ (a),(b): yes (c): no

$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b): yes (a),(c): no

Problem. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$, where $\mathbf{v}_0 = (3/5, 0, -4/5)$.

(a) Find the matrix B of the operator L.
(b) Find the range and kernel of L.
(c) Find the eigenvalues of L.
(d) Find the matrix of the operator L^{2017} (L applied 2017 times).

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \quad \mathbf{v}_0 = (3/5, 0, -4/5).$$

Let $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Then

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4/5 \\ y & z \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} 3/5 & -4/5 \\ x & z \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} 3/5 & 0 \\ x & y \end{vmatrix} \mathbf{e}_3$$

$$= \tfrac{4}{5}y\mathbf{e}_1 - \left(\tfrac{4}{5}x + \tfrac{3}{5}z\right)\mathbf{e}_2 + \tfrac{3}{5}y\mathbf{e}_3 = \left(\tfrac{4}{5}y, -\tfrac{4}{5}x - \tfrac{3}{5}z, \tfrac{3}{5}y\right).$$

In particular, $L(\mathbf{e}_1) = \left(0, -\frac{4}{5}, 0\right)$, $L(\mathbf{e}_2) = \left(\frac{4}{5}, 0, \frac{3}{5}\right)$, $L(\mathbf{e}_3) = \left(0, -\frac{3}{5}, 0\right)$.

Therefore $B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}$.

The range of the operator L is spanned by columns of the matrix B. It follows that $\mathrm{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0,1,0)$ and $\mathbf{v}_2 = (4,0,3)$.

The kernel of L is the nullspace of the matrix B, i.e., the solution set for the equation $B\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies x + \tfrac{3}{4}z = y = 0 \implies \mathbf{x} = t(-3/4, 0, 1).$$

Alternatively, the kernel of L is the set of vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$.
It follows that this is the line spanned by $\mathbf{v}_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix B:

$$\det(B - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

The eigenvalues are 0, i, and $-i$.

The matrix of the operator L^{2017} is B^{2017}.

Since the matrix B has eigenvalues 0, i, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $B = UDU^{-1}$, where U is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then $B^{2017} = UD^{2017}U^{-1}$. We have that $D^{2017} =$ $= \mathrm{diag}\big(0, i^{2017}, (-i)^{2017}\big) = \mathrm{diag}(0, i, -i) = D$.
Hence

$$B^{2017} = UDU^{-1} = B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}.$$