MATH 304
Linear Algebra

Lecture 2:
Gaussian elimination.
System of linear equations

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \quad \vdots & \quad \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

Here \(x_1, x_2, \ldots, x_n\) are variables and \(a_{ij}, b_j\) are constants.

A solution of the system is a common solution of all equations in the system.

A system of linear equations can have one solution, infinitely many solutions, or no solution at all.
\[
\begin{align*}
\begin{cases}
    x - y &= -2 \\
    2x + 3y &= 6 
\end{cases}
\end{align*}
\]

\[x = 0, \ y = 2\]
\[ \begin{align*}
2x + 3y &= 2 \\
2x + 3y &= 6
\end{align*} \]

\textit{inconsistent system} (no solutions)
\[\begin{align*}
4x + 6y &= 12 \\
2x + 3y &= 6
\end{align*}\] \[\iff\] \[2x + 3y = 6\]
Solving systems of linear equations

*Elimination method* always works for systems of linear equations.

*Algorithm:* (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

\[ \begin{align*}
x - y &= 2 \quad \Rightarrow \quad x = y + 2 \\
2x - y - z &= 5 \quad \Rightarrow \quad 2(y + 2) - y - z = 5
\end{align*} \]

After the elimination is completed, the system is solved by *back substitution*.

\[ \begin{align*}
y &= 1 \quad \Rightarrow \quad x = y + 2 = 3
\end{align*} \]
Gaussian elimination

Gaussian elimination is a modification of the elimination method that allows only so-called elementary operations.

Elementary operations for systems of linear equations:
(1) to multiply an equation by a nonzero scalar;
(2) to add an equation multiplied by a scalar to another equation;
(3) to interchange two equations.

Theorem Applying elementary operations to a system of linear equations does not change the solution set of the system.
Operation 1: multiply the $i$th equation by $r \neq 0$.

\[
\begin{align*}
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
\quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= b_i \\
\quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{cases}
\end{align*}
\]

\[
\Rightarrow \quad \begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
\quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
(ra_{i1})x_1 + (ra_{i2})x_2 + \cdots + (ra_{in})x_n &= rb_i \\
\quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{cases}
\]

To undo the operation, multiply the $i$th equation by $r^{-1}$. 
Operation 2: add $r$ times the $i$th equation to the $j$th equation.

\[
\begin{align*}
\{ & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\
\} & \quad \Rightarrow \\
\{ & a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\
\}
\end{align*}
\]

To undo the operation, add $-r$ times the $i$th equation to the $j$th equation.
Operation 3: interchange the $i$th and $j$th equations.

\[
\begin{align*}
\begin{cases}
\cdots \cdots \\
\quad a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\
\cdots \cdots
\end{cases}
\begin{cases}
\cdots \cdots \\
\quad a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\
\cdots \cdots
\end{cases}
\end{align*}
\]

To undo the operation, apply it once more.
Example.

\[
\begin{align*}
2x - y - z &= 3 \\
-x + y + z &= 6
\end{align*}
\]

Add \(-2\) times the 1st equation to the 2nd equation:

\[
\begin{align*}
x - y &= 2 \\
y - z &= -1 \\
x + y + z &= 6
\end{align*}
\]

\[
R2 := R2 - 2 \times R1
\]

Add \(-1\) times the 1st equation to the 3rd equation:

\[
\begin{align*}
x - y &= 2 \\
y - z &= -1 \\
2y + z &= 4
\end{align*}
\]
Add $-2$ times the 2nd equation to the 3rd equation:

\[
\begin{align*}
x - y &= 2 \\
y - z &= -1 \\
3z &= 6
\end{align*}
\]

The elimination is completed, and we can solve the system by back substitution. However we may as well proceed with elementary operations.

Multiply the 3rd equation by $1/3$:

\[
\begin{align*}
x - y &= 2 \\
y - z &= -1 \\
z &= 2
\end{align*}
\]
Add the 3rd equation to the 2nd equation:

\[
\begin{align*}
    x - y &= 2 \\
    y &= 1 \\
    z &= 2
\end{align*}
\]

Add the 2nd equation to the 1st equation:

\[
\begin{align*}
    x &= 3 \\
    y &= 1 \\
    z &= 2
\end{align*}
\]
System of linear equations:

\[
\begin{align*}
x - y & = 2 \\
2x - y - z & = 3 \\
x + y + z & = 6
\end{align*}
\]

Solution: \((x, y, z) = (3, 1, 2)\)
Another example.

\[
\begin{align*}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  -x + 4y - 3z &= 1
\end{align*}
\]

Add the 1st equation to the 3rd equation:

\[
\begin{align*}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  5y - 5z &= 2
\end{align*}
\]

Add \(-5\) times the 2nd equation to the 3rd equation:

\[
\begin{align*}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  0 &= -13
\end{align*}
\]
System of linear equations:

\[
\begin{align*}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  -x + 4y - 3z &= 1 \\
\end{align*}
\]

**Solution:** no solution (*inconsistent system*).
Yet another example.

\[
\begin{align*}
x + y - 2z &= 1 \\
y - z &= 3 \\
-x + 4y - 3z &= 14
\end{align*}
\]

Add the 1st equation to the 3rd equation:

\[
\begin{align*}
x + y - 2z &= 1 \\
y - z &= 3 \\
5y - 5z &= 15
\end{align*}
\]

Add $-5$ times the 2nd equation to the 3rd equation:

\[
\begin{align*}
x + y - 2z &= 1 \\
y - z &= 3 \\
0 &= 0
\end{align*}
\]
Add $-1$ times the 2nd equation to the 1st equation:

\[
\begin{align*}
\begin{cases}
x - z &= -2 \\
y - z &= 3 \\
0 &= 0
\end{cases}
\end{align*}
\]

$\iff$

\[
\begin{align*}
\begin{cases}
x &= z - 2 \\
y &= z + 3
\end{cases}
\end{align*}
\]

Here $z$ is a free variable.

It follows that

\[
\begin{align*}
\begin{cases}
x &= t - 2 \\
y &= t + 3
\end{cases}
\end{align*}
\]

for some $t \in \mathbb{R}$.
System of linear equations:

\[
\begin{align*}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  -x + 4y - 3z &= 14
\end{align*}
\]

Solution: \((x, y, z) = (t - 2, t + 3, t), \quad t \in \mathbb{R}.
\]

In vector form, \((x, y, z) = (-2, 3, 0) + t(1, 1, 1)\).

The set of all solutions is a line in \(\mathbb{R}^3\) passing through the point \((-2, 3, 0)\) in the direction \((1, 1, 1)\).
Matrices

**Definition.** A *matrix* is a rectangular array of numbers.

Examples:

\[
\begin{pmatrix}
2 & 7 \\
-1 & 0 \\
3 & 3
\end{pmatrix}, \quad \begin{pmatrix}
2 & 7 & 0.2 \\
4.6 & 1 & 1
\end{pmatrix}, \\
\begin{pmatrix}
3/5 \\
5/8 \\
4
\end{pmatrix}, \quad (\sqrt{2}, 0, -\sqrt{3}, 5), \quad \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
\]

**dimensions** = (# of rows) × (# of columns)

*n*-by-*n*: square matrix

*n*-by-1: column vector

1-by-*n*: row vector
System of linear equations:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    &\quad \hdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

Coefficient matrix and column vector of the right-hand sides:

\[
\begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_m
\end{pmatrix}
\]
System of linear equations:

$$\begin{cases} 
\begin{align*} 
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m 
\end{align*} 
\end{cases}$$

Augmented matrix:

$$\begin{pmatrix} 
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m 
\end{pmatrix}$$
Elementary operations for systems of linear equations correspond to *elementary row operations* for augmented matrices:

1. to multiply a row by a nonzero scalar;
2. to add the $i$th row multiplied by some $r \in \mathbb{R}$ to the $j$th row;
3. to interchange two rows.

*Remark.* Rows are added and multiplied by scalars as vectors (namely, row vectors).