Lecture 38: Orthogonal polynomials.
**Problem.** Approximate the function \( f(x) = e^x \) on the interval \([-1, 1]\) by a quadratic polynomial.

The best approximation would be a polynomial \( p(x) \) that minimizes the distance relative to the uniform norm:

\[
\| f - p \|_\infty = \max_{|x| \leq 1} |f(x) - p(x)|.
\]

However there is no analytic way to find such a polynomial. Another approach is to find a "least squares" approximation that minimizes the integral norm

\[
\| f - p \|_2 = \left( \int_{-1}^{1} |f(x) - p(x)|^2 \, dx \right)^{1/2}.
\]
The norm $\| \cdot \|_2$ is induced by the inner product
\[ \langle g, h \rangle = \int_{-1}^{1} g(x)h(x) \, dx. \]

Therefore $\| f - p \|_2$ is minimal if $p$ is the orthogonal projection of the function $f$ on the subspace $\mathcal{P}_3$ of quadratic polynomials.

Suppose that $p_0, p_1, p_2$ is an orthogonal basis for $\mathcal{P}_3$. Then
\[ p(x) = \frac{\langle f, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) + \frac{\langle f, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x) + \frac{\langle f, p_2 \rangle}{\langle p_2, p_2 \rangle} p_2(x). \]
Orthogonal polynomials

\( \mathcal{P} \): the vector space of all polynomials with real coefficients:
\[ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \]
Basis for \( \mathcal{P} \): \( 1, x, x^2, \ldots, x^n, \ldots \)

Suppose that \( \mathcal{P} \) is endowed with an inner product.

**Definition.** Orthogonal polynomials (relative to the inner product) are polynomials \( p_0, p_1, p_2, \ldots \) such that \( \deg p_n = n \) (\( p_0 \) is a nonzero constant) and \( \langle p_n, p_m \rangle = 0 \) for \( n \neq m \).
Orthogonal polynomials can be obtained by applying the Gram-Schmidt orthogonalization process to the basis $1, x, x^2, \ldots$:

$p_0(x) = 1,$

$p_1(x) = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x),$

$p_2(x) = x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x),$

\[ \cdots \]

$p_n(x) = x^n - \frac{\langle x^n, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \cdots - \frac{\langle x^n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1}(x),$

\[ \cdots \]

Then $p_0, p_1, p_2, \ldots$ are orthogonal polynomials.
**Theorem (a)** Orthogonal polynomials always exist. **(b)** The orthogonal polynomial of a fixed degree is unique up to scaling. 

**(c)** A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, q \rangle = 0$ for any polynomial $q$ with $\deg q < \deg p$. 

**(d)** A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, x^k \rangle = 0$ for any $0 \leq k < \deg p$. 
Example. \[ \langle p, q \rangle = \int_{-1}^{1} p(x)q(x) \, dx. \]

Note that \( \langle x^n, x^m \rangle = 0 \) if \( m + n \) is odd.

Hence \( p_{2k}(x) \) contains only even powers of \( x \) while \( p_{2k+1}(x) \) contains only odd powers of \( x \).

\[
\begin{align*}
p_0(x) &= 1, \\
p_1(x) &= x, \\
p_2(x) &= x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} = x^2 - \frac{1}{3}, \\
p_3(x) &= x^3 - \frac{\langle x^3, x \rangle}{\langle x, x \rangle}x = x^3 - \frac{3}{5}x.
\end{align*}
\]

\( p_0, p_1, p_2, \ldots \) are called the **Legendre polynomials**.
Instead of normalization, the orthogonal polynomials are subject to **standardization**.

The standardization for the Legendre polynomials is $P_n(1) = 1$. In particular, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

**Problem.** Find $P_4(x)$.

Let $P_4(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$.

We know that $P_4(1) = 1$ and $\langle P_4, x^k \rangle = 0$ for $0 \leq k \leq 3$.

$P_4(1) = a_4 + a_3 + a_2 + a_1 + a_0$,

$\langle P_4, 1 \rangle = \frac{2}{5} a_4 + \frac{2}{3} a_2 + 2a_0$, $\langle P_4, x \rangle = \frac{2}{5} a_3 + \frac{2}{3} a_1$,

$\langle P_4, x^2 \rangle = \frac{2}{7} a_4 + \frac{2}{5} a_2 + \frac{2}{3} a_0$, $\langle P_4, x^3 \rangle = \frac{2}{7} a_3 + \frac{2}{5} a_1$. 
\[
\begin{align*}
\{ &a_4 + a_3 + a_2 + a_1 + a_0 = 1 \\
&\frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\
&\frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\
&\frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \\
&\frac{2}{7}a_3 + \frac{2}{5}a_1 = 0
\end{align*}
\]

\[
\begin{align*}
\{ &\frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\
&\frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \quad \Rightarrow \quad a_1 = a_3 = 0
\end{align*}
\]

\[
\begin{align*}
\{ &a_4 + a_2 + a_0 = 1 \\
&\frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \quad \Leftrightarrow \quad \begin{cases} a_4 = \frac{35}{8} \\ a_2 = -\frac{30}{8} \\ a_0 = \frac{3}{8} \end{cases}
\end{align*}
\]

Thus \( P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \).
Legendre polynomials
Problem. Find a quadratic polynomial that is the best least squares fit to the function \( f(x) = |x| \) on the interval \([-1, 1]\).

The best least squares fit is a polynomial \( p(x) \) that minimizes the distance relative to the integral norm

\[
\| f - p \| = \left( \int_{-1}^{1} |f(x) - p(x)|^2 \, dx \right)^{1/2}
\]

over all polynomials of degree 2.

The norm \( \| f - p \| \) is minimal if \( p \) is the orthogonal projection of the function \( f \) on the subspace \( \mathcal{P}_3 \) of polynomials of degree at most 2.
The Legendre polynomials \( P_0, P_1, P_2 \) form an orthogonal basis for \( \mathcal{P}_3 \). Therefore

\[
p(x) = \frac{\langle f, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) + \frac{\langle f, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1(x) + \frac{\langle f, P_2 \rangle}{\langle P_2, P_2 \rangle} P_2(x).
\]

\[
\langle f, P_0 \rangle = \int_{-1}^{1} |x| \, dx = 2 \int_{0}^{1} x \, dx = 1,
\]

\[
\langle f, P_1 \rangle = \int_{-1}^{1} |x| \, x \, dx = 0,
\]

\[
\langle f, P_2 \rangle = \int_{-1}^{1} |x| \frac{3x^2 - 1}{2} \, dx = \int_{0}^{1} x(3x^2 - 1) \, dx = \frac{1}{4},
\]

\[
\langle P_0, P_0 \rangle = \int_{-1}^{1} dx = 2, \quad \langle P_2, P_2 \rangle = \int_{-1}^{1} \left( \frac{3x^2 - 1}{2} \right)^2 \, dx = \frac{2}{5}.
\]

In general, \( \langle P_n, P_n \rangle = \frac{2}{2n + 1} \).
Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

Solution: $p(x) = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x) = \frac{1}{2} + \frac{5}{16}(3x^2 - 1) = \frac{3}{16}(5x^2 + 1)$.

Recurrent formula for the Legendre polynomials:

$(n + 1)P_{n+1} = (2n + 1)xP_n(x) - nP_{n-1}(x)$.

For example, $4P_4(x) = 7xP_3(x) - 3P_2(x)$. 
Definition. Chebyshev polynomials $T_0, T_1, T_2, \ldots$ are orthogonal polynomials relative to the inner product

$$
\langle p, q \rangle = \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1-x^2}} \, dx,
$$

with the standardization $T_n(1) = 1$.

Remark. “T” is like in “Tschebyscheff”.

Change of variable in the integral: $x = \cos \phi$.

$$
\langle p, q \rangle = -\int_{0}^{\pi} \frac{p(\cos \phi)q(\cos \phi)}{\sqrt{1 - \cos^2 \phi}} \cos' \phi \, d\phi
\quad = \int_{0}^{\pi} p(\cos \phi)q(\cos \phi) \, d\phi.
$$
Theorem. \[ T_n(\cos \phi) = \cos n\phi. \]

\[
\langle T_n, T_m \rangle = \int_0^\pi T_n(\cos \phi) T_m(\cos \phi) \, d\phi
\]

\[
= \int_0^\pi \cos(n\phi) \cos(m\phi) \, d\phi = 0 \text{ if } n \neq m.
\]

Recurrent formula: \[ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \]

\[ T_0(x) = 1, \quad T_1(x) = x, \]
\[ T_2(x) = 2x^2 - 1, \]
\[ T_3(x) = 4x^3 - 3x, \]
\[ T_4(x) = 8x^4 - 8x^2 + 1, \ldots \]

That is, \[ \cos 2\phi = 2\cos^2 \phi - 1, \]
\[ \cos 3\phi = 4\cos^3 \phi - 3\cos \phi, \]
\[ \cos 4\phi = 8\cos^4 \phi - 8\cos^2 \phi + 1, \ldots \]
Chebyshev polynomials