MATH 304
Linear Algebra

Lecture 18:
Rank and nullity of a matrix.
Let $A = (a_{ij})$ be an $m \times n$ matrix.

**Definition.** The **nullspace** of the matrix $A$, denoted $N(A)$, is the set of all $n$-dimensional column vectors $x$ such that $Ax = 0$.

$$
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
=
\begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}
$$

The nullspace $N(A)$ is the solution set of a system of linear homogeneous equations (with $A$ as the coefficient matrix).
Let $A = (a_{ij})$ be an $m \times n$ matrix.

**Theorem** The nullspace $N(A)$ is a subspace of the vector space $\mathbb{R}^n$.

**Proof:** We have to show that $N(A)$ is nonempty, closed under addition, and closed under scaling.

First of all, $A\mathbf{0} = \mathbf{0} \implies \mathbf{0} \in N(A) \implies N(A)$ is not empty.

Secondly, if $\mathbf{x}, \mathbf{y} \in N(A)$, i.e., if $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$, then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0} \implies \mathbf{x} + \mathbf{y} \in N(A)$.

Thirdly, if $\mathbf{x} \in N(A)$, i.e., if $A\mathbf{x} = \mathbf{0}$, then for any $r \in \mathbb{R}$ one has $A(r\mathbf{x}) = r(A\mathbf{x}) = r\mathbf{0} = \mathbf{0} \implies r\mathbf{x} \in N(A)$.

**Definition.** The dimension of the nullspace $N(A)$ is called the **nullity** of the matrix $A$. 
Problem. Find the nullity of the matrix

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}. \]

Elementary row operations do not change the nullspace. Let us convert \( A \) to reduced row echelon form:

\[
\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}
\]

\[
\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{cases}
\]

General element of \( N(A) \):

\[
(x_1, x_2, x_3, x_4) = (t + 2s, -2t - 3s, t, s) = t(1, -2, 1, 0) + s(2, -3, 0, 1), \quad t, s \in \mathbb{R}.
\]

Vectors \((1, -2, 1, 0)\) and \((2, -3, 0, 1)\) form a basis for \( N(A) \). Thus the nullity of the matrix \( A \) is 2.
Row space

*Definition.* The **row space** of an \( m \times n \) matrix \( A \) is the subspace of \( \mathbb{R}^n \) spanned by rows of \( A \). The dimension of the row space is called the **rank** of the matrix \( A \).

**Theorem 1**  Elementary row operations do not change the row space of a matrix.

**Theorem 2**  If a matrix \( A \) is in row echelon form, then the nonzero rows of \( A \) are linearly independent.

**Corollary**  The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

**Theorem 3**  The rank of a matrix \( A \) plus the nullity of \( A \) equals the number of columns of \( A \).
Problem. Find the rank of the matrix

\[
A = \begin{pmatrix}
-1 & 0 & -1 & 2 \\
2 & 0 & 2 & 0 \\
1 & 0 & 1 & -1
\end{pmatrix}.
\]

Elementary row operations do not change the row space. Let us convert \( A \) to row echelon form:

\[
\begin{pmatrix}
-1 & 0 & -1 & 2 \\
2 & 0 & 2 & 0 \\
1 & 0 & 1 & -1
\end{pmatrix} \rightarrow \begin{pmatrix}
-1 & 0 & -1 & 2 \\
0 & 0 & 0 & 4 \\
1 & 0 & 1 & -1
\end{pmatrix}
\]

\[
\rightarrow \begin{pmatrix}
-1 & 0 & -1 & 2 \\
0 & 0 & 0 & 4 \\
0 & 0 & 0 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
-1 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Vectors \((1, 0, 1, -2)\) and \((0, 0, 0, 1)\) form a basis for the row space of \(A\). Thus the rank of \(A\) is 2.

*Remark.* The rank of \(A\) equals the number of nonzero rows in the row echelon form, which equals the number of leading entries.

The nullity of \(A\) equals the number of free variables in the corresponding system, which equals the number of columns without leading entries.

Consequently, \(\text{rank} + \text{nullity}\) is the number of all columns in the matrix \(A\).
**Theorem 1**  Elementary row operations do not change the row space of a matrix.

**Proof:** Suppose that $A$ and $B$ are $m \times n$ matrices such that $B$ is obtained from $A$ by an elementary row operation. Let $a_1, \ldots, a_m$ be the rows of $A$ and $b_1, \ldots, b_m$ be the rows of $B$. We have to show that $\text{Span}(a_1, \ldots, a_m) = \text{Span}(b_1, \ldots, b_m)$.

Observe that any row $b_i$ of $B$ belongs to $\text{Span}(a_1, \ldots, a_m)$. Indeed, either $b_i = a_j$ for some $1 \leq j \leq m$, or $b_i = ra_i$ for some scalar $r \neq 0$, or $b_i = a_i + ra_j$ for some $j \neq i$ and $r \in \mathbb{R}$.

It follows that $\text{Span}(b_1, \ldots, b_m) \subset \text{Span}(a_1, \ldots, a_m)$.

Now the matrix $A$ can also be obtained from $B$ by an elementary row operation. By the above,

$$\text{Span}(a_1, \ldots, a_m) \subset \text{Span}(b_1, \ldots, b_m).$$
Problem. Find the nullity of the matrix

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}. \]

Alternative solution: Clearly, the rows of \( A \) are linearly independent. Therefore the rank of \( A \) is 2. Since

\[ \text{(rank of } A) + \text{(nullity of } A) = 4, \]

it follows that the nullity of \( A \) is 2.
Definition. The **column space** of an $m \times n$ matrix $A$ is the subspace of $\mathbb{R}^m$ spanned by columns of $A$.

**Theorem 1** The column space of a matrix $A$ coincides with the row space of the transpose matrix $A^T$.

**Theorem 2** Elementary column operations do not change the column space of a matrix.

**Theorem 3** Elementary row operations do not change the dimension of the column space of a matrix (although they can change the column space).

**Theorem 4** For any matrix, the row space and the column space have the same dimension.
Problem. Find a basis for the column space of the matrix

\[ B = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{pmatrix}. \]

The column space of \( B \) coincides with the row space of \( B^T \). To find a basis, we convert \( B^T \) to row echelon form:

\[
\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \\ 2 & 3 & -2 \end{pmatrix}
\]
Thus vectors $(1, 2, -1)$ and $(0, 1, 0)$ form a basis for the column space of the matrix $B$. 
Problem. Find a basis for the column space of the matrix

\[ B = \begin{pmatrix}
1 & 0 & -1 & 2 \\
2 & 1 & 2 & 3 \\
-1 & 0 & 1 & -2
\end{pmatrix}. \]

Alternative solution: The dimension of the column space equals the dimension of the row space, which is 2 (since the first two rows are not parallel and the third row is a multiple of the first one).

The 1st and the 2nd columns, \((1, 2, -1)\) and \((0, 1, 0)\), are linearly independent. It follows that they form a basis for the column space (actually, any two columns form such a basis).