Lecture 29: Orthogonal sets. The Gram-Schmidt process.
Orthogonal sets

Let $V$ be an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$.

**Definition.** A nonempty set $S \subseteq V$ of nonzero vectors is called an **orthogonal set** if all vectors in $S$ are mutually orthogonal. That is, $0 \notin S$ and $\langle x, y \rangle = 0$ for any $x, y \in S$, $x \neq y$.

An orthogonal set $S \subseteq V$ is called **orthonormal** if $\| x \| = 1$ for any $x \in S$.

**Remark.** Vectors $v_1, v_2, \ldots, v_k \in V$ form an orthonormal set if and only if

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
Examples. • \( V = \mathbb{R}^n, \langle x, y \rangle = x \cdot y \).

The standard basis \( e_1 = (1, 0, 0, \ldots, 0) \),
\( e_2 = (0, 1, 0, \ldots, 0) \), \ldots, \( e_n = (0, 0, 0, \ldots, 1) \).
It is an orthonormal set.

• \( V = \mathbb{R}^3, \langle x, y \rangle = x \cdot y \).
\( v_1 = (3, 5, 4), \ v_2 = (3, -5, 4), \ v_3 = (4, 0, -3) \).
\( v_1 \cdot v_2 = 0, \ v_1 \cdot v_3 = 0, \ v_2 \cdot v_3 = 0, \)
\( v_1 \cdot v_1 = 50, \ v_2 \cdot v_2 = 50, \ v_3 \cdot v_3 = 25. \)

Thus the set \( \{v_1, v_2, v_3\} \) is orthogonal but not orthonormal. An orthonormal set is formed by normalized vectors \( w_1 = \frac{v_1}{\|v_1\|}, \ w_2 = \frac{v_2}{\|v_2\|}, \)
\( w_3 = \frac{v_3}{\|v_3\|}. \)
• \( V = C[-\pi, \pi], \quad \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx. \)

\( f_1(x) = \sin x, \quad f_2(x) = \sin 2x, \ldots, \quad f_n(x) = \sin nx, \ldots \)

\[
\langle f_m, f_n \rangle = \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx
\]

\[= \int_{-\pi}^{\pi} \frac{1}{2} \left( \cos(mx - nx) - \cos(mx + nx) \right) \, dx.\]

\[
\int_{-\pi}^{\pi} \cos(kx) \, dx = \frac{\sin(kx)}{k} \bigg|_{x=-\pi}^{\pi} = 0 \quad \text{if} \quad k \in \mathbb{Z}, \quad k \neq 0.
\]

\[k = 0 \quad \implies \quad \int_{-\pi}^{\pi} \cos(kx) \, dx = \int_{-\pi}^{\pi} \, dx = 2\pi.\]
\[ \langle f_m, f_n \rangle = \frac{1}{2} \int_{-\pi}^{\pi} \left( \cos(m - n)x - \cos(m + n)x \right) dx \]

\[ = \begin{cases} 
\pi & \text{if } m = n \\
0 & \text{if } m \neq n 
\end{cases} \]

Thus the set \( \{f_1, f_2, f_3, \ldots \} \) is orthogonal but not orthonormal.

It is orthonormal with respect to a scaled inner product

\[ \langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx. \]
Orthogonality $\iff$ linear independence

**Theorem**  Suppose $v_1, v_2, \ldots, v_k$ are nonzero vectors that form an orthogonal set. Then $v_1, v_2, \ldots, v_k$ are linearly independent.

**Proof:**  Suppose $t_1v_1 + t_2v_2 + \cdots + t_kv_k = 0$ for some $t_1, t_2, \ldots, t_k \in \mathbb{R}$.

Then for any index $1 \leq i \leq k$ we have

$$\langle t_1v_1 + t_2v_2 + \cdots + t_kv_k, v_i \rangle = \langle 0, v_i \rangle = 0.$$ 

$$\implies t_1\langle v_1, v_i \rangle + t_2\langle v_2, v_i \rangle + \cdots + t_k\langle v_k, v_i \rangle = 0$$

By orthogonality, $t_i\langle v_i, v_i \rangle = 0 \implies t_i = 0$. 
Orthonormal bases

Let \( v_1, v_2, \ldots, v_n \) be an orthonormal basis for an inner product space \( V \).

**Theorem** Let \( x = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \) and \( y = y_1 v_1 + y_2 v_2 + \cdots + y_n v_n \), where \( x_i, y_j \in \mathbb{R} \). Then

1. \( \langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \),
2. \( \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \).

**Proof:** (ii) follows from (i) when \( y = x \).

\[
\langle x, y \rangle = \left\langle \sum_{i=1}^{n} x_i v_i, \sum_{j=1}^{n} y_j v_j \right\rangle = \sum_{i=1}^{n} x_i \left\langle v_i, \sum_{j=1}^{n} y_j v_j \right\rangle
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \langle v_i, v_j \rangle = \sum_{i=1}^{n} x_i y_i.
\]
Let $v_1, v_2, \ldots, v_n$ be a basis for an inner product space $V$.

**Theorem** If the basis $v_1, v_2, \ldots, v_n$ is an orthogonal set then for any $x \in V$

$$x = \frac{\langle x, v_1 \rangle}{\langle v_1, v_1 \rangle}v_1 + \frac{\langle x, v_2 \rangle}{\langle v_2, v_2 \rangle}v_2 + \cdots + \frac{\langle x, v_n \rangle}{\langle v_n, v_n \rangle}v_n.$$

If $v_1, v_2, \ldots, v_n$ is an orthonormal set then

$$x = \langle x, v_1 \rangle v_1 + \langle x, v_2 \rangle v_2 + \cdots + \langle x, v_n \rangle v_n.$$

**Proof:** We have that $x = x_1v_1 + \cdots + x_nv_n$.

$$\implies \langle x, v_i \rangle = \langle x_1v_1 + \cdots + x_nv_n, v_i \rangle, \quad 1 \leq i \leq n.$$  

$$\implies \langle x, v_i \rangle = x_1\langle v_1, v_i \rangle + \cdots + x_n\langle v_n, v_i \rangle$$

$$\implies \langle x, v_i \rangle = x_i\langle v_i, v_i \rangle.$$
Let $V$ be a vector space with an inner product. Suppose that $v_1, \ldots, v_k \in V$ are nonzero vectors that form an orthogonal set. Given $x \in V$, let
\[
p = \frac{\langle x, v_1 \rangle}{\langle v_1, v_1 \rangle}v_1 + \cdots + \frac{\langle x, v_k \rangle}{\langle v_k, v_k \rangle}v_k, \quad o = x - p.
\]
Let $W$ denote the span of $v_1, \ldots, v_k$.

**Theorem (a)** $o \perp w$ for all $w \in W$ (denoted $o \perp W$).

(b) $\|o\| = \|x - p\| = \min_{w \in W} \|x - w\|$.

Thus $p$ is the **orthogonal projection** of the vector $x$ on the subspace $W$. Also, $p$ is closer to $x$ than any other vector in $W$, and $\|o\| = \text{dist}(x, p)$ is the **distance** from $x$ to $W$. 
Orthogonalization

Let $V$ be a vector space with an inner product. Suppose $x_1, x_2, \ldots, x_n$ is a basis for $V$. Let

$v_1 = x_1,$

$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1,$

$v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle x_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2,$

\ldots

$v_n = x_n - \frac{\langle x_n, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \cdots - \frac{\langle x_n, v_{n-1} \rangle}{\langle v_{n-1}, v_{n-1} \rangle} v_{n-1}.$

Then $v_1, v_2, \ldots, v_n$ is an orthogonal basis for $V$. The orthogonalization of a basis as described above is called the Gram-Schmidt process.
span of $x_1, x_2$
Normalization

Let $V$ be a vector space with an inner product. Suppose $v_1, v_2, \ldots, v_n$ is an orthogonal basis for $V$. Let $w_1 = \frac{v_1}{\|v_1\|}, w_2 = \frac{v_2}{\|v_2\|}, \ldots, w_n = \frac{v_n}{\|v_n\|}$. Then $w_1, w_2, \ldots, w_n$ is an orthonormal basis for $V$.

**Theorem**  Any finite-dimensional vector space with an inner product has an orthonormal basis.

*Remark.*  An infinite-dimensional vector space with an inner product may or may not have an orthonormal basis.