MATH 304
Linear Algebra

Lecture 33:
Bases of eigenvectors.
Diagonalization.
**Eigenvalues and eigenvectors of an operator**

*Definition.* Let $V$ be a vector space and $L : V \to V$ be a linear operator. A number $\lambda$ is called an **eigenvalue** of the operator $L$ if $L(v) = \lambda v$ for a nonzero vector $v \in V$. The vector $v$ is called an **eigenvector** of $L$ associated with the eigenvalue $\lambda$. (If $V$ is a functional space then eigenvectors are also called **eigenfunctions**.)

If $V = \mathbb{R}^n$ then the linear operator $L$ is given by $L(x) = Ax$, where $A$ is an $n \times n$ matrix.

In this case, eigenvalues and eigenvectors of the operator $L$ are precisely eigenvalues and eigenvectors of the matrix $A$. 
**Theorem**  If $v_1, v_2, \ldots, v_k$ are eigenvectors of a linear operator $L$ associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $v_1, v_2, \ldots, v_k$ are linearly independent.

**Corollary**  Let $A$ be an $n \times n$ matrix such that the characteristic equation $\det(A - \lambda I) = 0$ has $n$ distinct real roots. Then $\mathbb{R}^n$ has a basis consisting of eigenvectors of $A$.

*Proof:*  Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct real roots of the characteristic equation. Any $\lambda_i$ is an eigenvalue of $A$, hence there is an associated eigenvector $v_i$. By the theorem, vectors $v_1, v_2, \ldots, v_n$ are linearly independent. Therefore they form a basis for $\mathbb{R}^n$. 
**Theorem**  If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct real numbers, then the functions $e^{\lambda_1x}, e^{\lambda_2x}, \ldots, e^{\lambda_kx}$ are linearly independent.

**Proof:** Consider a linear operator $D : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ given by $Df = f'$. Then $e^{\lambda_1x}, \ldots, e^{\lambda_kx}$ are eigenfunctions of $D$ associated with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. 
Characteristic polynomial of an operator

Let $L$ be a linear operator on a finite-dimensional vector space $V$. Let $u_1, u_2, \ldots, u_n$ be a basis for $V$. Let $A$ be the matrix of $L$ with respect to this basis.

**Definition.** The characteristic polynomial of the matrix $A$ is called the characteristic polynomial of the operator $L$.

Then eigenvalues of $L$ are roots of its characteristic polynomial.

**Theorem.** The characteristic polynomial of the operator $L$ is well defined. That is, it does not depend on the choice of a basis.
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Proof: Let $B$ be the matrix of $L$ with respect to a different basis $v_1, v_2, \ldots, v_n$. Then $A = UBU^{-1}$, where $U$ is the transition matrix from the basis $v_1, \ldots, v_n$ to $u_1, \ldots, u_n$. We have to show that $\det(A - \lambda I) = \det(B - \lambda I)$ for all $\lambda \in \mathbb{R}$. We obtain

$$
\det(A - \lambda I) = \det(UBU^{-1} - \lambda I) \\
= \det(UBU^{-1} - U(\lambda I)U^{-1}) = \det(U(B - \lambda I)U^{-1}) \\
= \det(U) \det(B - \lambda I) \det(U^{-1}) = \det(B - \lambda I).
$$
Diagonalization

Let $L$ be a linear operator on a finite-dimensional vector space $V$. Then the following conditions are equivalent:

- the matrix of $L$ with respect to some basis is diagonal;
- there exists a basis for $V$ formed by eigenvectors of $L$.

The operator $L$ is **diagonalizable** if it satisfies these conditions.

Let $A$ be an $n \times n$ matrix. Then the following conditions are equivalent:

- $A$ is the matrix of a diagonalizable operator;
- $A$ is similar to a diagonal matrix, i.e., it is represented as $A = UBU^{-1}$, where the matrix $B$ is diagonal;
- there exists a basis for $\mathbb{R}^n$ formed by eigenvectors of $A$.

The matrix $A$ is **diagonalizable** if it satisfies these conditions. Otherwise $A$ is called **defective**.
Example. \[ A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \]

- The matrix \( A \) has two eigenvalues: 1 and 3.
- The eigenspace of \( A \) associated with the eigenvalue 1 is the line spanned by \( \mathbf{v}_1 = (-1, 1) \).
- The eigenspace of \( A \) associated with the eigenvalue 3 is the line spanned by \( \mathbf{v}_2 = (1, 1) \).
- Eigenvectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) form a basis for \( \mathbb{R}^2 \).

Thus the matrix \( A \) is diagonalizable. Namely, 
\[ A = U B U^{-1}, \]
where 
\[ B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \]
Example. \( A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \).

- The matrix \( A \) has two eigenvalues: 0 and 2.
- The eigenspace corresponding to 0 is spanned by \( \mathbf{v}_1 = (-1, 1, 0) \).
- The eigenspace corresponding to 2 is spanned by \( \mathbf{v}_2 = (1, 1, 0) \) and \( \mathbf{v}_3 = (-1, 0, 1) \).
- Eigenvectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) form a basis for \( \mathbb{R}^3 \).

Thus the matrix \( A \) is diagonalizable. Namely,

\[
A = UBU^{-1},
\]

where

\[
B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Problem. Diagonalize the matrix $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$.

We need to find a diagonal matrix $B$ and an invertible matrix $U$ such that $A = UBU^{-1}$.

Suppose that $\mathbf{v}_1 = (x_1, y_1)$, $\mathbf{v}_2 = (x_2, y_2)$ is a basis for $\mathbb{R}^2$ formed by eigenvectors of $A$, i.e., $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for some $\lambda_i \in \mathbb{R}$. Then we can take

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad U = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$  

Note that $U$ is the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2$ to the standard basis.
Problem. Diagonalize the matrix $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$.

Characteristic equation of $A$: $\begin{vmatrix} 4 - \lambda & 3 \\ 0 & 1 - \lambda \end{vmatrix} = 0$.

$(4 - \lambda)(1 - \lambda) = 0 \implies \lambda_1 = 4, \; \lambda_2 = 1$.

Associated eigenvectors: $v_1 = (1, 0), \; v_2 = (-1, 1)$.

Thus $A = U B U^{-1}$, where

$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \; U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. 
Problem. Let \( A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} \). Find \( A^5 \).

We know that \( A = UBU^{-1} \), where

\[
B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.
\]

Then \( A^5 = UBU^{-1} UBU^{-1} UBU^{-1} UBU^{-1} UBU^{-1} \)

\[
= UB^5 U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1024 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1024 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1024 & 1023 \\ 0 & 1 \end{pmatrix}.
\]
Problem. Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find a matrix $C$ such that $C^2 = A$.

We know that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. $$

Suppose that $D^2 = B$ for some matrix $D$. Let $C = UDU^{-1}$. Then $C^2 = UDU^{-1} UDU^{-1} = UD^2 U^{-1} = UBU^{-1} = A$.

We can take $D = \begin{pmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. 
There are *two obstructions* to existence of a basis consisting of eigenvectors. They are illustrated by the following examples.

*Example 1.* $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\det(A - \lambda I) = (\lambda - 1)^2$. Hence $\lambda = 1$ is the only eigenvalue. The associated eigenspace is the line $t(1, 0)$.

*Example 2.* $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = \lambda^2 + 1$.

$\implies$ no real eigenvalues or eigenvectors

(However there are *complex* eigenvalues/eigenvectors.)