MATH 311–504 May 7, 2008

Final exam: Solutions

Problem 1 (15 pts.) Find a quadratic polynomial $p(x)$ such that $p(-2) = 7$ and $p(1) = p'(1) = 1$.

Solution: $p(x) = x^2 - x + 1$.

Let $p(x) = ax^2 + bx + c$, where a, b, c are unknown coefficients. Then $p(-2) = 4a - 2b + c$, $p(1) = a + b + c$, and $p'(1) = 2a + b$. The coefficients a, b, and c are to be chosen so that

$$\begin{cases} 4a - 2b + c = 7, \\ a + b + c = 1, \\ 2a + b = 1. \end{cases}$$

Solving this system of linear equations, we obtain that

$$\begin{cases} 4a - 2b + c = 7 \\ a + b + c = 1 \\ 2a + b = 1 \end{cases} \iff \begin{cases} 3a - 3b = 6 \\ a + b + c = 1 \\ 2a + b = 1 \end{cases} \iff \begin{cases} a - b = 2 \\ a + b + c = 1 \\ 2a + b = 1 \end{cases}$$

$$\iff \begin{cases} a - b = 2 \\ a + b + c = 1 \\ 3a = 3 \end{cases} \iff \begin{cases} a - b = 2 \\ a + b + c = 1 \\ a = 1 \end{cases} \iff \begin{cases} b = -1 \\ a + b + c = 1 \\ a = 1 \end{cases} \iff \begin{cases} a = 1 \\ b = -1 \\ c = 1 \end{cases}$$

Thus $p(x) = x^2 - x + 1$.

Problem 2 (20 pts.) Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2 + (\mathbf{v} \cdot \mathbf{v}_3)\mathbf{v}_4,$$

where $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (1, 2, 0)$, $\mathbf{v}_3 = (-2, 1, 2)$, $\mathbf{v}_4 = (1, 1, 1)$.

(i) Find the matrix of the operator L.

Solution: $\begin{pmatrix} -1 & 1 & 3 \\ 0 & 1 & 4 \\ -2 & 1 & 2 \end{pmatrix}$.

Let A denote the matrix of the linear operator L. The columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$, where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ is the standard basis for $\mathbb{R}^3$. We obtain

$$L(\mathbf{e}_1) = (\mathbf{e}_1 \cdot \mathbf{v}_1)\mathbf{v}_2 + (\mathbf{e}_1 \cdot \mathbf{v}_3)\mathbf{v}_4 = \mathbf{v}_2 - 2\mathbf{v}_4 = (-1, 0, -2),$$
$$L(\mathbf{e}_2) = (\mathbf{e}_2 \cdot \mathbf{v}_1)\mathbf{v}_2 + (\mathbf{e}_2 \cdot \mathbf{v}_3)\mathbf{v}_4 = \mathbf{v}_4 = (1, 1, 1),$$
$$L(\mathbf{e}_3) = (\mathbf{e}_3 \cdot \mathbf{v}_1)\mathbf{v}_2 + (\mathbf{e}_3 \cdot \mathbf{v}_3)\mathbf{v}_4 = \mathbf{v}_2 + 2\mathbf{v}_4 = (3, 4, 2).$$

Therefore

$$A = \begin{pmatrix} -1 & 1 & 3 \\ 0 & 1 & 4 \\ -2 & 1 & 2 \end{pmatrix}.$$

(ii) Find the dimension of the image of L.

Solution: 2.

The image of the operator L is spanned by columns of its matrix, that is, by vectors $\mathbf{x}_1 = (-1, 0, -2)$, $\mathbf{x}_2 = (1, 1, 1)$, and $\mathbf{x}_3 = (3, 4, 2)$. The third column is a linear combination of the first two: $\mathbf{x}_3 = \mathbf{x}_1 + 4\mathbf{x}_2$. It follows that the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ alone span the image of L. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, they form a basis for the image.

(iii) Find a basis for the null-space of L.

Solution: $(-1, -4, 1)$.

The null-space of the operator L is the set of vectors $\mathbf{x} = (x, y, z)$ such that $L(\mathbf{x}) = \mathbf{0}$. Since the vectors $\mathbf{v}_2$ and $\mathbf{v}_4$ are linearly independent, this is equivalent to $\mathbf{x} \cdot \mathbf{v}_1 = \mathbf{x} \cdot \mathbf{v}_3 = 0$. Solving the latter system of equations, we obtain

$$\begin{cases} \mathbf{x} \cdot \mathbf{v}_1 = 0 \\ \mathbf{x} \cdot \mathbf{v}_3 = 0 \end{cases} \iff \begin{cases} x + z = 0 \\ -2x + y + 2z = 0 \end{cases} \iff \begin{cases} x + z = 0 \\ y + 4z = 0 \end{cases}$$

The general solution is $x = -t$, $y = -4t$, $z = t$, where $t \in \mathbb{R}$. Hence the null-space of L is the line $t(-1, -4, 1)$. The vector $(-1, -4, 1)$ is a basis for this line.

Problem 3 (20 pts.) Let $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

Solution: $\det A = 1$.

The determinant of A is easily evaluated using row or column expansions. For example, let us expand the determinant by the first row:

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{vmatrix} = - \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}.$$

Then expand the 3-by-3 determinant by the first column:

$$\det A = - \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} = 1.$$

Another way to evaluate $\det A$ is to reduce the matrix A to the identity matrix using elementary row operations (see below). This requires more work but we are going to do it anyway, to find the inverse of A.

(ii) Find the inverse matrix A^{-1}.

Solution: $A^{-1} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$

First we merge the matrix A with the identity matrix into one 4-by-8 matrix:

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Interchange the first row with the third row:

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Add the second row to the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & 1 \end{array}\right).$$

Interchange the third row with the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

Multiply both the second and the third row by -1:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

Add the fourth row to the third row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

Add the fourth row to the first row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

Subtract the third row from the first row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

Finally the left part of our 4-by-8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

As a byproduct, we can evaluate the determinant of A. We have transformed A into the identity matrix using elementary row operations. These included two row exchanges and two row multiplications, both times by -1. It follows that $\det I = (-1)^2 \det A$. Thus $\det A = \det I = 1$.

Problem 4 (25 pts.) Let $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

Solution: 0, 1, 2.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. One obtains that

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\big((1-\lambda)^2 - 1\big) = -\lambda(1-\lambda)(2-\lambda).$$

Hence the matrix B has three eigenvalues: 0, 1, and 2.

(ii) For each eigenvalue of B, find an associated eigenvector.

Solution: $(1, -1, 1)$, $(0, 0, 1)$, and $(1, 1, 1)$ are eigenvectors of B associated with the eigenvalues 0, 1, and 2, respectively.

An eigenvector $\mathbf{x} = (x, y, z)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{x} = \mathbf{0}$. First consider the case $\lambda = 0$. We obtain that

$$B\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)$ is an eigenvector of B associated with the eigenvalue 0.

Next consider the case $\lambda = 1$. We obtain that

$$(B - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x = 0, \\ y = 0. \end{cases}$$

The general solution is $x = 0$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (0, 0, 1)$ is an eigenvector of B associated with the eigenvalue 1.

Finally, consider the case $\lambda = 2$. We obtain that

$$(B - 2I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = t$, $y = t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of B associated with the eigenvalue 2.

(iii) Find a diagonal matrix Λ and an invertible matrix U such that $B = U\Lambda U^{-1}$.

Solution: $\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

The vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (0, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B accosiated with distinct eigenvalues 0, 1, and 2, respectively. Therefore these vectors are linearly independent, which implies that they form a basis for $\mathbb{R}^3$. It follows that $B = U\Lambda U^{-1}$, where

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Here U is the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (its columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$) while Λ is the matrix of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{x}) = B\mathbf{x}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Problem 5 (20 pts.) Let V be a three-dimensional subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, 0, 0, 1)$, $\mathbf{x}_2 = (0, 1, 1, 2)$, and $\mathbf{x}_3 = (2, 1, 3, 2)$.

(i) Find an orthogonal basis for V.

Solution: $\mathbf{v}_1 = (1, 0, 0, 1)$, $\mathbf{v}_2 = (-1, 1, 1, 1)$, $\mathbf{v}_3 = (1, 0, 2, -1)$.

To find an orthogonal basis for the subspace V, we apply the Gram-Schmidt orthogonalization process to the spanning set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 = (1, 0, 0, 1), \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (0, 1, 1, 2) - \frac{2}{2}(1, 0, 0, 1) = (-1, 1, 1, 1), \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (2, 1, 3, 2) - \frac{4}{2}(1, 0, 0, 1) - \frac{4}{4}(-1, 1, 1, 1) = (1, 0, 2, -1).
\end{aligned}$$

By construction, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are orthogonal to each other. It follows that they are linearly independent. Also, the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is the same as the span of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, that is, the subspace V. Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal basis for V.

(ii) Find the distance from the point $\mathbf{y} = (4, 4, 2, 2)$ to the subspace V.

Solution: $2\sqrt{3}$.

The distance from $\mathbf{y}$ to V is the distance from $\mathbf{y}$ to the closest point $\mathbf{y}_0$ in V, which is the orthogonal projection of $\mathbf{y}$ on V. Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal basis for V, it follows that

$$\mathbf{y}_0 = \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{y}\cdot\mathbf{v}_3}{\mathbf{v}_3\cdot\mathbf{v}_3}\mathbf{v}_3$$

$$= \frac{6}{2}(1, 0, 0, 1) + \frac{4}{4}(-1, 1, 1, 1) + \frac{6}{6}(1, 0, 2, -1) = (3, 1, 3, 3).$$

Then $\mathbf{y}-\mathbf{y}_0 = (4, 4, 2, 2)-(3, 1, 3, 3) = (1, 3, -1, -1)$ and the desired distance is $|\mathbf{y}-\mathbf{y}_0| = \sqrt{12} = 2\sqrt{3}$.

Bonus Problem 6 (15 pts.) Find the area of the triangle in $\mathbb{R}^2$ bounded by the lines $x + y = 1$, $x - 2y = 1$, and $2x - y = -1$.

Solution: $\frac{3}{2}$.

Let $\mathbf{x}_1 = (x_1, y_1)$ be the intersection point of the lines $x + y = 1$ and $x - 2y = 1$. Let $\mathbf{x}_2 = (x_2, y_2)$ be the intersection point of the lines $x + y = 1$ and $2x - y = -1$. Let $\mathbf{x}_3 = (x_3, y_3)$ be the intersection point of the lines $x - 2y = 1$ and $2x - y = -1$.

The points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ are vertices of the triangle. Their coordinates can be found from the following systems of linear equations:

$$\begin{cases} x_1 + y_1 = 1, \\ x_1 - 2y_1 = 1; \end{cases} \qquad \begin{cases} x_2 + y_2 = 1, \\ 2x_2 - y_2 = -1; \end{cases} \qquad \begin{cases} x_3 - 2y_3 = 1, \\ 2x_3 - y_3 = -1. \end{cases}$$

Solving the systems, we obtain that $\mathbf{x}_1 = (1, 0)$, $\mathbf{x}_2 = (0, 1)$, and $\mathbf{x}_3 = (-1, -1)$.

The vectors $\mathbf{w}_1 = \mathbf{x}_2 - \mathbf{x}_1 = (-1, 1)$ and $\mathbf{w}_2 = \mathbf{x}_3 - \mathbf{x}_1 = (-2, -1)$ are represented by adjacent sides of the triangle. Therefore the area a of the triangle is equal to

$$a = \frac{1}{2}\,|\mathbf{w}_1|\,|\mathbf{w}_2|\sin\alpha,$$

where α is the angle between vectors $\mathbf{w}_1$ and $\mathbf{w}_2$. We have

$$\cos\alpha = \frac{\mathbf{w}_1\cdot\mathbf{w}_2}{|\mathbf{w}_1|\,|\mathbf{w}_2|} = \frac{1}{\sqrt{2}\,\sqrt{5}} = \frac{1}{\sqrt{10}}.$$

Then $\sin\alpha = \sqrt{1-\cos^2\alpha} = 3/\sqrt{10}$. Finally,

$$a = \frac{1}{2}\,|\mathbf{w}_1|\,|\mathbf{w}_2|\sin\alpha = \frac{1}{2}\,\sqrt{2}\,\sqrt{5}\,\frac{3}{\sqrt{10}} = \frac{3}{2}.$$

Bonus Problem 7 (20 pts.) Let R denote a linear operator on $\mathbb{R}^3$ that permutes vectors from the standard basis as follows: $R(\mathbf{e}_1) = \mathbf{e}_2$, $R(\mathbf{e}_2) = \mathbf{e}_3$, $R(\mathbf{e}_3) = \mathbf{e}_1$.

(i) Explain why R is a rotation.

Let C be the matrix of the linear operator R. The operator is a rotation if and only if the matrix C is orthogonal and $\det C = 1$. We have that

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Clearly, the columns of C form an orthonormal set. Hence C is orthogonal. Also, it is easy to check that $\det C = 1$.

(ii) Find the axis of the rotation R.

Solution: $t(1, 1, 1)$.

The axis of rotation consists of points $\mathbf{x} = (x, y, z)$ such that $R(\mathbf{x}) = \mathbf{x}$. We obtain that

$$R(\mathbf{x}) = \mathbf{x} \iff (C - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution of the latter system is $x = t$, $y = t$, $z = t$, where $t \in \mathbb{R}$. Thus the axis of the rotation R is the line $t(1, 1, 1)$.

(iii) Find the angle of the rotation R.

Solution: $\dfrac{2\pi}{3}$.

Let ϕ denote the angle of rotation. We may assume that $0 \leq \phi \leq \pi$. Then the operator R^3 is a rotation by angle 3ϕ. We obtain that $R^3(\mathbf{e}_1) = \mathbf{e}_1$, $R^3(\mathbf{e}_2) = \mathbf{e}_2$, $R^3(\mathbf{e}_3) = \mathbf{e}_3$. It follows that R^3 is the identity. Therefore 3ϕ must be a multiple of 2π. We have $0 \leq 3\phi \leq 3\pi$. Note that $\phi \neq 0$ since R itself is not the identity. Thus $3\phi = 2\pi$ and $\phi = 2\pi/3$.