MATH 311-504

Topics in Applied Mathematics

Lecture 12:
Evaluation of determinants.
Cross product.

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ij})_{1 \le i,j \le n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A = 0$ if and only if the matrix A is not invertible.

Definition in low dimensions

Definition. $\det(a) = a, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+: \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$-: \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Properties of determinants

Determinants and elementary row operations:

- if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;
- if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;
- if we interchange two rows of a matrix, the determinant changes its sign.

Properties of determinants

Tests for non-invertibility:

- if a matrix A has a zero row then $\det A = 0$;
- if a matrix A has two identical rows then $\det A = 0$;
- if a matrix has two proportional rows then $\det A = 0$.

Properties of determinants

Special matrices:

- $\det I = 1$;
- the determinant of a diagonal matrix is equal to the product of its diagonal entries;
- the determinant of an upper triangular matrix is equal to the product of its diagonal entries.

Properties of determinants

Determinant of the transpose:

- If A is a square matrix then $\det A^T = \det A$.

Columns vs. rows:

- if one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar;
- adding a scalar multiple of one column to another does not change the determinant;
- interchanging two columns of a matrix changes the sign of its determinant;
- if a matrix A has a zero column or two proportional columns then $\det A = 0$.

Properties of determinants

Determinants and matrix multiplication:

- if A and B are $n \times n$ matrices then
$$\det(AB) = \det A \cdot \det B;$$
- if A and B are $n \times n$ matrices then
$$\det(AB) = \det(BA);$$
- if A is an invertible matrix then
$$\det(A^{-1}) = (\det A)^{-1}.$$

Determinants and scalar multiplication:

- if A is an $n \times n$ matrix and $r \in \mathbb{R}$ then
$$\det(rA) = r^n \det A.$$

Examples

$$X = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -2 & 1 \end{pmatrix}.$$

$$\det X = (-1)\cdot 2\cdot(-3) = 6, \quad \det Y = \det Y^T = 3,$$

$$\det(XY) = 6\cdot 3 = 18, \quad \det(YX) = 3\cdot 6 = 18,$$

$$\det(Y^{-1}) = 1/3, \quad \det(XY^{-1}) = 6/3 = 2,$$

$$\det(XYX^{-1}) = \det Y = 3, \quad \det(X^{-1}Y^{-1}XY) = 1,$$

$$\det(2X) = 2^3 \det X = 2^3\cdot 6 = 48,$$

$$\det(-3X^{-1}Y) = (-3)^3\cdot 6^{-1}\cdot 3 = -27/2.$$

Row and column expansions

Given an $n \times n$ matrix $A = (a_{ij})$, let A_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the ith row and the jth column of A.

Theorem For any $1 \leq k, m \leq n$ we have that

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det A_{kj},$$

(*expansion by kth row*)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det A_{im}.$$

(*expansion by mth column*)

Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Expansion by the 1st row:

$$\begin{pmatrix} \boxed{1} & * & * \\ * & 5 & 6 \\ * & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} * & * & \boxed{3} \\ 4 & 5 & * \\ 7 & 8 & * \end{pmatrix}$$

$$\det A = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0.$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Expansion by the 2nd column:

$$\begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ * & \boxed{5} & * \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ 4 & * & 6 \\ * & \boxed{8} & * \end{pmatrix}$$

$$\det A = -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$$= -2(4 \cdot 9 - 6 \cdot 7) + 5(1 \cdot 9 - 3 \cdot 7) - 8(1 \cdot 6 - 3 \cdot 4) = 0.$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Subtract the 1st row from the 2nd row and from the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$$

since the last matrix has two proportional rows.

Another example. $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{pmatrix}$.

First let's do some row reduction.

Add -4 times the 1st row to the 2nd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix}$$

Add -7 times the 1st row to the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

Expand the determinant by the 1st column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix}$$

Thus

$$\det B = \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix} = (-3) \begin{vmatrix} 1 & 2 \\ -6 & -8 \end{vmatrix}$$

$$= (-3)(-2) \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (-3)(-2)(-2) = -12$$

Determinants and linear dependence

Theorem For any n-by-n matrix A the following conditions are equivalent:

- $\det A = 0$;
- A is not invertible;
- the matrix equation $A\mathbf{x} = \mathbf{0}$ has a nonzero solution $\mathbf{x} \in \mathbb{R}^n$;
- columns of A are linearly dependent vectors;
- rows of A are linearly dependent vectors.

Problem. Determine whether vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (4, 5, 6)$, and $\mathbf{v}_3 = (7, 8, 13)$ are parallel to the same plane.

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are parallel to the same plane if and only if they are linearly dependent.

Consider a 3×3 matrix V whose rows are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. We have

$$\det V = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix} = -12.$$

$\det V \neq 0 \implies$ the vectors are linearly independent $\implies$ the vectors are not parallel to the same plane

Cross product

Definition. Given two vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ in $\mathbb{R}^3$, the **cross product** $\mathbf{v} \times \mathbf{w}$ is another vector in $\mathbb{R}^3$ defined by

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1).$$

Using the standard basis $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$, the definition can be rewritten as

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\mathbf{i} + (v_3 w_1 - v_1 w_3)\mathbf{j} + (v_1 w_2 - v_2 w_1)\mathbf{k}$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Proposition For any vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ in $\mathbb{R}^3$,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Indeed, since

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k},$$

it follows that

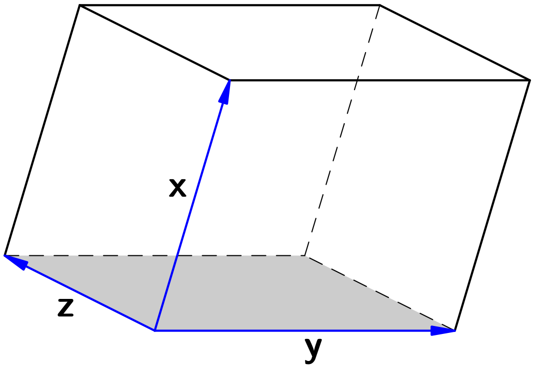
$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}.$$

Algebraic properties of the cross product:

- $\mathbf{x} \times \mathbf{y}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$
- $\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y}$ (anticommutativity)
- $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z}$ (distributive law)
- $r(\mathbf{x} \times \mathbf{y}) = (r\mathbf{x}) \times \mathbf{y} = \mathbf{x} \times (r\mathbf{y})$
- $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$
- In general, $(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} \neq \mathbf{x} \times (\mathbf{y} \times \mathbf{z})$
- $(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} + (\mathbf{y} \times \mathbf{z}) \times \mathbf{x} + (\mathbf{z} \times \mathbf{x}) \times \mathbf{y} = \mathbf{0}$
  (Jacobi's identity)
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

Geometric properties of the cross product:

- $\mathbf{x} \times \mathbf{y}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$.
- If $\mathbf{x} \times \mathbf{y} \neq \mathbf{0}$ then the triple of vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x} \times \mathbf{y}$ obeys the same rule (right-hand or left-hand rule) as the standard basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$.
- The area of the parallelogram with vectors $\mathbf{x}$ and $\mathbf{y}$ as adjacent sides is equal to $|\mathbf{x} \times \mathbf{y}|$. That is, $|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}|\,|\mathbf{y}| \sin \angle(\mathbf{x}, \mathbf{y})$.
- The area of the triangle with vectors $\mathbf{x}$ and $\mathbf{y}$ as adjacent sides is equal to $\frac{1}{2}|\mathbf{x} \times \mathbf{y}|$.
- The volume of the parallelepiped with vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ as adjacent edges is equal to $|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$.



Area of the grey parallelogram $= |\mathbf{y} \times \mathbf{z}|$.

Volume of the parallelepiped $= |\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$.

The triple $\mathbf{x}, \mathbf{y}, \mathbf{z}$ obeys the right-hand rule.

Problem. (i) Find volume of the parallelepiped with vectors $\mathbf{a} = (1, 4, 7)$, $\mathbf{b} = (2, 5, 8)$, and $\mathbf{c} = (3, 6, 13)$ as adjacent edges.

(ii) Determine whether the triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$ obeys the same hand rule as the standard basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix} = -12.$$

Volume of the parallelepiped $= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 12.$

Since $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) < 0$, the triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$ does not obey the same hand rule as the triple $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

Suppose Π is a plane in $\mathbb{R}^3$ with a parametric representation $t_1\mathbf{v} + t_2\mathbf{w} + \mathbf{u}$, where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$.

Recall that $\mathbf{u}$ is a point in Π while $\mathbf{v}$ and $\mathbf{w}$ are vectors parallel to the plane. Then the vector $\mathbf{v} \times \mathbf{w}$ is orthogonal to the plane.

Therefore the plane Π is given by the equation $(\mathbf{x} - \mathbf{u}) \cdot (\mathbf{v} \times \mathbf{w}) = 0$ or

$$\begin{vmatrix} x - u_1 & y - u_2 & z - u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0,$$

where $\mathbf{x} = (x, y, z)$.