MATH 311-504

Topics in Applied Mathematics

Lecture 2-11:
Eigenvalues and eigenvectors (continued).
Bases of eigenvectors.

Eigenvalues and eigenvectors

Definition. Let V be a vector space and $L : V \to V$ be a linear operator. A number λ is called an **eigenvalue** of the operator L if $\boxed{L(\mathbf{v}) = \lambda\mathbf{v}}$ for a *nonzero* vector $\mathbf{v} \in V$. The vector $\mathbf{v}$ is called an **eigenvector** of L associated with the eigenvalue λ.

Eigenvalues and eigenvectors of a matrix transformation $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$ are also called eigenvalues and eigenvectors of the matrix A.

Eigenspaces

Let $L : V \to V$ be a linear operator. For any $\lambda \in \mathbb{R}$ let V_λ denotes the set of all solutions of the equation $L(\mathbf{x}) = \lambda\mathbf{x}$.

V_λ is a *subspace* of V since V_λ is the *null-space* of the linear operator $\mathbf{x} \mapsto L(\mathbf{x}) - \lambda\mathbf{x}$.

V_λ consists of all eigenvectors of L associated with the eigenvalue λ plus the zero vector. In particular, λ is an eigenvalue of L if and only if $V_\lambda \neq \{\mathbf{0}\}$.

If $V_\lambda \neq \{\mathbf{0}\}$ then it is called the **eigenspace** of L associated with the eigenvalue λ.

Examples. • $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}),\ \ D(f) = f'$.

A nonzero function $f \in C^\infty(\mathbb{R})$ is an eigenfunction of the operator D associated with an eigenvalue λ if $f'(x) = \lambda f(x)$ for all $x \in \mathbb{R}$. That is, if $f(x) = ce^{\lambda x}$, where c is a nonzero constant.

Thus each $\lambda \in \mathbb{R}$ is an eigenvalue of D.
The corresponding eigenspace is spanned by $e^{\lambda x}$.

• $D_0 : \mathcal{P} \to \mathcal{P},\ \ D_0(p) = p'$.

The only eigenvalue of D_0 is 0. The corresponding eigenspace consists of costants.

Eigenvalues and eigenvectors of a matrix

Let A be an n-by-n matrix and $\mathbf{x} \in \mathbb{R}^n$ be a column vector. Then $A\mathbf{x} = \lambda\mathbf{x} \iff (A - \lambda I)\mathbf{x} = \mathbf{0}$.

λ is an eigenvalue $\iff$ the matrix $A - \lambda I$ is not invertible $\iff$ $\det(A - \lambda I) = 0$

Definition. $\det(A - \lambda I) = 0$ is called the **characteristic equation** of the matrix A.

Eigenvalues λ of A are roots of the characteristic equation. Associated eigenvectors of A are nonzero solutions of the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Example. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

$$= (a - \lambda)(d - \lambda) - bc$$

$$= \lambda^2 - (a + d)\lambda + (ad - bc).$$

Example. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

$$= -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3,$$

where $c_1 = a_{11} + a_{22} + a_{33}$ (the trace of A),

$c_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$

$c_3 = \det A.$

Theorem. Let $A = (a_{ij})$ be an n-by-n matrix. Then $\det(A - \lambda I)$ is a polynomial of λ of degree n:

$$\det(A - \lambda I) = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n.$$

Furthermore, $(-1)^{n-1} c_1 = a_{11} + a_{22} + \cdots + a_{nn}$ and $c_n = \det A$.

Corollary Any n-by-n matrix has at most n eigenvalues.

Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 1 and 3.

- The eigenspace of A associated with the eigenvalue 1 is the line $t(-1,1)$.

- The eigenspace of A associated with the eigenvalue 3 is the line $t(1,1)$.

- Eigenvectors $\mathbf{v}_1 = (-1,1)$ and $\mathbf{v}_2 = (1,1)$ of the matrix A form a basis for $\mathbb{R}^2$.

- Geometrically, the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is a stretch by a factor of 3 away from the line $x + y = 0$ in the orthogonal direction.

Example. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Characteristic equation:

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0.$$

Expand the determinant by the 3rd row:

$$(2-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0.$$

$\big((1-\lambda)^2 - 1\big)(2-\lambda) = 0 \iff -\lambda(2-\lambda)^2 = 0$
$\implies \quad \lambda_1 = 0, \;\; \lambda_2 = 2.$

$$A\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Convert the matrix to reduced form:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{0} \iff \begin{cases} x + y = 0, \\ z = 0. \end{cases}$$

The general solution is $(-t, t, 0) = t(-1, 1, 0)$, $t \in \mathbb{R}$. Thus $\mathbf{v}_1 = (-1, 1, 0)$ is an eigenvector associated with the eigenvalue 0. The corresponding eigenspace is the line spanned by $\mathbf{v}_1$.

$$(A - 2I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x - y + z = 0.$$

The general solution is $x = t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. Equivalently,

$$\mathbf{x} = (t - s, t, s) = t(1,1,0) + s(-1,0,1).$$

Thus $\mathbf{v}_2 = (1,1,0)$ and $\mathbf{v}_3 = (-1,0,1)$ are eigenvectors associated with the eigenvalue 2. The corresponding eigenspace is the plane spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$.

Summary. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 0 and 2.
- The eigenvalue 0 is *simple:* the associated eigenspace is a line.
- The eigenvalue 2 is of *multiplicity* 2: the associated eigenspace is a plane.
- Eigenvectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (-1, 0, 1)$ of the matrix A form a basis for $\mathbb{R}^3$.
- Geometrically, the map $\mathbf{x} \mapsto A\mathbf{x}$ is the projection on the plane $\mathrm{Span}(\mathbf{v}_2, \mathbf{v}_3)$ along lines parallel to $\mathbf{v}_1$ with the subsequent scaling by a factor of 2.

Systems of linear ODEs

Basis consisting of eigenvectors of a matrix is useful when solving systems of linear ODEs with constant coefficients.

Example.
$$\begin{cases} \frac{dx}{dt} = x + y - z, \\ \frac{dy}{dt} = x + y + z, \\ \frac{dz}{dt} = 2z. \end{cases}$$

Let $\mathbf{v} = (x, y, z)$. Then the system can be rewritten in vector form

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Vectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (-1, 0, 1)$ form a basis for $\mathbb{R}^3$.

Therefore the vector-function $\mathbf{v}(t)$ is uniquely represented as $\mathbf{v}(t) = r_1(t)\mathbf{v}_1 + r_2(t)\mathbf{v}_2 + r_3(t)\mathbf{v}_3$, where $r_1(t)$, $r_2(t)$, and $r_3(t)$ are scalar functions.

$$\frac{d\mathbf{v}}{dt} = \frac{dr_1}{dt}\mathbf{v}_1 + \frac{dr_2}{dt}\mathbf{v}_2 + \frac{dr_3}{dt}\mathbf{v}_3, \qquad A\mathbf{v} = 2r_2\mathbf{v}_2 + 2r_3\mathbf{v}_3.$$

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v} \quad \Longleftrightarrow \quad \begin{cases} \frac{dr_1}{dt} = 0, \\ \frac{dr_2}{dt} = 2r_2, \\ \frac{dr_3}{dt} = 2r_3. \end{cases}$$

The general solution: $r_1(t) = c_1$, $r_2(t) = c_2 e^{2t}$, $r_3(t) = c_3 e^{2t}$, where c_1, c_2, c_3 are arbitrary constants.

Thus $\mathbf{v}(t) = r_1(t)\mathbf{v}_1 + r_2(t)\mathbf{v}_2 + r_3(t)\mathbf{v}_3 =$
$= c_1(-1,1,0) + c_2 e^{2t}(1,1,0) + c_3 e^{2t}(-1,0,1).$

System:
$$\begin{cases} \frac{dx}{dt} = x + y - z, \\ \frac{dy}{dt} = x + y + z, \\ \frac{dz}{dt} = 2z. \end{cases}$$

Solution:
$$\begin{cases} x(t) = -c_1 + (c_2 - c_3)e^{2t}, \\ y(t) = c_1 + c_2 e^{2t}, \\ z(t) = c_3 e^{2t}. \end{cases}$$

Theorem If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are eigenvectors of a linear operator L associated with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent.

Proof in the case $k = 2$: Assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. Then $\mathbf{v}_1 = t\mathbf{v}_2$ for some $t \in \mathbb{R}$. It follows that $L(\mathbf{v}_1) = \lambda_2 \mathbf{v}_1$. But $L(\mathbf{v}_1) = \lambda_1 \mathbf{v}_1 \implies \lambda_1 \mathbf{v}_1 = \lambda_2 \mathbf{v}_1$ $\implies (\lambda_1 - \lambda_2)\mathbf{v}_1 = \mathbf{0} \implies \mathbf{v}_1 = \mathbf{0}$, a contradiction.

Proof in the case $k = 3$: Suppose that $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3 = \mathbf{0}$ for some $t_1, t_2, t_3 \in \mathbb{R}$. Then $L(t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3) = \mathbf{0}$

$$\implies t_1 L(\mathbf{v}_1) + t_2 L(\mathbf{v}_2) + t_3 L(\mathbf{v}_3) = \mathbf{0}$$

$$\implies t_1\lambda_1\mathbf{v}_1 + t_2\lambda_2\mathbf{v}_2 + t_3\lambda_3\mathbf{v}_3 = \mathbf{0}.$$

Subtract λ_3 times the first equality from the last equality:

$$t_1(\lambda_1 - \lambda_3)\mathbf{v}_1 + t_2(\lambda_2 - \lambda_3)\mathbf{v}_2 = \mathbf{0}.$$

By the above $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Therefore $t_1(\lambda_1 - \lambda_3) = t_2(\lambda_2 - \lambda_3) = 0 \implies t_1 = t_2 = 0 \implies t_3 = 0.$

Corollary 1 Suppose A is an $n \times n$ matrix that has n distinct eigenvalues. Then $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.

Proof: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues of A. Any λ_i has an associated eigenvector $\mathbf{v}_i$. By the theorem, vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent. Therefore they form a basis for $\mathbb{R}^n$.

Corollary 2 If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct real numbers, then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}$ are linearly independent.

Proof: Consider a linear operator $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by $D(f) = f'$. We have that $e^{\lambda_1 x}, \ldots, e^{\lambda_k x}$ are eigenfunctions of D associated with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.