MATH 311-504

Topics in Applied Mathematics

Lecture 2-2: Linear maps (continued). Matrix transformations.

Vector space

Vector space is a set V equipped with two operations $\alpha : V \times V \to V$ and $\mu : \mathbb{R} \times V \to V$ that have certain properties (listed below).

The operation α is called *addition*. For any $\mathbf{u}, \mathbf{v} \in V$, the element $\alpha(\mathbf{u}, \mathbf{v})$ is denoted $\mathbf{u} + \mathbf{v}$.

The operation μ is called *scalar multiplication*. For any $r \in \mathbb{R}$ and $\mathbf{u} \in V$, the element $\mu(r, \mathbf{u})$ is denoted $r\mathbf{u}$.

Properties of addition and scalar multiplication

A1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ for all $\mathbf{a},\mathbf{b}\in V$.

A2. $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ for all $\mathbf{a},\mathbf{b},\mathbf{c}\in V$.

A3. There exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$ for all $\mathbf{a}\in V$.

A4. For any $\mathbf{a}\in V$ there exists an element of V, denoted $-\mathbf{a}$, such that $\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}$.

A5. $r(\mathbf{a}+\mathbf{b})=r\mathbf{a}+r\mathbf{b}$ for all $r\in\mathbb{R}$ and $\mathbf{a},\mathbf{b}\in V$.

A6. $(r+s)\mathbf{a}=r\mathbf{a}+s\mathbf{a}$ for all $r,s\in\mathbb{R}$ and $\mathbf{a}\in V$.

A7. $(rs)\mathbf{a}=r(s\mathbf{a})$ for all $r,s\in\mathbb{R}$ and $\mathbf{a}\in V$.

A8. $1\mathbf{a}=\mathbf{a}$ for all $\mathbf{a}\in V$.

Examples of vector spaces

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m \times n$ matrices with real entries
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \dots)$, $x_i \in \mathbb{R}$
- $\{\mathbf{0}\}$: the trivial vector space
- $F(\mathbb{R})$: the set of all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$
- $C^1(\mathbb{R})$: all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$
- $C^\infty(\mathbb{R})$: all smooth functions $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Remark. A function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$ is a linear transformation of the vector space $\mathbb{R}$ if and only if $b = 0$.

Properties of linear mappings

Let $L : V_1 \to V_2$ be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$

for all $k \geq 1$, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V_1$, and $r_1, \ldots, r_k \in \mathbb{R}$.

$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2)$,
$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) =$
$= r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2) + r_3L(\mathbf{v}_3)$, and so on.

- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.

$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2$.

- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v})$.

Examples of linear mappings

- *Scaling* $L : V \to V, \; L(\mathbf{v}) = s\mathbf{v}$, where $s \in \mathbb{R}$.

- *Dot product with a fixed vector*
$\ell : \mathbb{R}^n \to \mathbb{R}, \; \ell(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^n$.

- *Cross product with a fixed vector*
$L : \mathbb{R}^3 \to \mathbb{R}^3, \; L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^3$.

- *Multiplication by a fixed matrix*
$L : \mathbb{R}^n \to \mathbb{R}^m, \; L(\mathbf{v}) = A\mathbf{v}$, where A is an $m \times n$ matrix and all vectors are column vectors.

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$\ell : F(\mathbb{R}) \to \mathbb{R}$, $\ell(f) = f(a)$, where $a \in \mathbb{R}$.

- *Multiplication by a fixed function*

$L : F(\mathbb{R}) \to F(\mathbb{R})$, $L(f) = gf$, where $g \in F(\mathbb{R})$.

- *Differentiation* $D : C^1(\mathbb{R}) \to C(\mathbb{R})$, $L(f) = f'$.

- *Integration over a finite interval*

$\ell : C(\mathbb{R}) \to \mathbb{R}$, $\ell(f) = \int_a^b f(x)\,dx$, where

$a, b \in \mathbb{R}$, $a < b$.

Theorem Let V_1, V_2, V_3 be vector spaces. If mappings $f : V_1 \to V_2$ and $g : V_2 \to V_3$ are linear then their composition $h = g \circ f : V_1 \to V_3$ given by $h(\mathbf{v}) = g(f(\mathbf{v}))$ is also linear.

$$h(\mathbf{x}+\mathbf{y}) = g(f(\mathbf{x}+\mathbf{y})) = g(f(\mathbf{x}) + f(\mathbf{y})) =$$
$$= g(f(\mathbf{x})) + g(f(\mathbf{y})) = h(\mathbf{x}) + h(\mathbf{y}),$$
$$h(r\mathbf{x}) = g(f(r\mathbf{x})) = g(rf(\mathbf{x})) = rg(f(\mathbf{x})) = rh(\mathbf{x}).$$

Examples. • $\ell : C^1(\mathbb{R}) \to \mathbb{R},\ \ell(f) = f'(a).$

• $\ell : C[a,b] \to \mathbb{R},\ \ell(f) = \displaystyle\int_a^b g(x)f(x)\,dx.$

• *Integral operator* $L : C[a,b] \to C[a,b]$,

$$\big(L(f)\big)(x) = \int_a^b G(x,y)\,f(y)\,dy.$$

Linear differential operators

- an ordinary differential operator

$$L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on $\mathbb{R}$.
That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- Laplace's operator $\Delta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Matrix transformations

Any $m \times n$ matrix A gives rise to a transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{x}) = A\mathbf{x}$, where all vectors are regarded as column vectors. This transformation is **linear**.

Example. $$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis for $\mathbb{R}^3$. We have that

$$L(\mathbf{e}_1) = (1, 3, 0), \quad L(\mathbf{e}_2) = (0, 4, 5), \quad L(\mathbf{e}_3) = (2, 7, 8).$$

Thus $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$ are columns of the matrix.

Problem. Find a linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(\mathbf{e}_1) = (1,1)$, $L(\mathbf{e}_2) = (0,-2)$, $L(\mathbf{e}_3) = (3,0)$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for $\mathbb{R}^3$.

$$L(x,y,z) = L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3)$$
$$= xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3)$$
$$= x(1,1) + y(0,-2) + z(3,0) = (x+3z, x-2y)$$

$$L(x,y,z) = \begin{pmatrix} x+3z \\ x-2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Columns of the matrix are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$.

Theorem Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then there exists an $m\times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. The columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for $\mathbb{R}^n$.

Problem Find a linear mapping $L : \mathbb{R}^2 \to \mathbb{R}^3$ such that $L(2,1) = (1,2,1)$ and $L(3,1) = (0,1,1)$.

Let $\mathbf{v}_1 = (2,1)$, $\mathbf{v}_2 = (3,1)$. Then $\mathbf{e}_1 = \mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{e}_2 = 3\mathbf{v}_1 - 2\mathbf{v}_2$. Since L is linear, it follows that

$$L(\mathbf{e}_1) = L(\mathbf{v}_2) - L(\mathbf{v}_1) = (0,1,1) - (1,2,1) = (-1,-1,0),$$
$$L(\mathbf{e}_2) = 3L(\mathbf{v}_1) - 2L(\mathbf{v}_2) = 3(1,2,1) - 2(0,1,1) = (3,4,1).$$

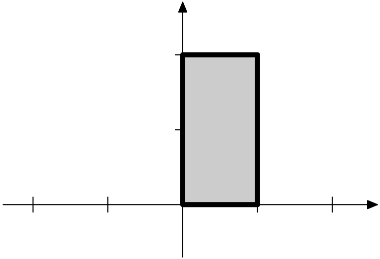
Thus $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$

Linear transformations of $\mathbb{R}^2$

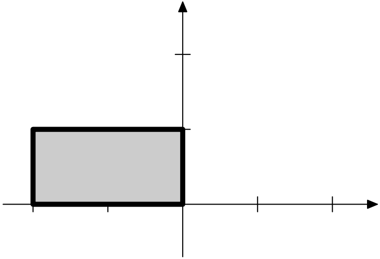
Any linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is represented as multiplication of a 2-dimensional column vector by a $2{\times}2$ matrix: $f(\mathbf{x}) = A\mathbf{x}$ or

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

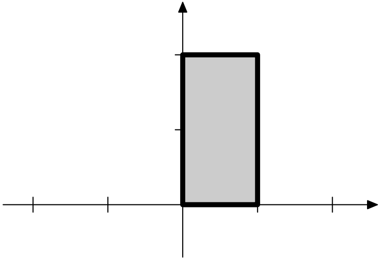
Linear transformations corresponding to different matrices can have various geometric properties.



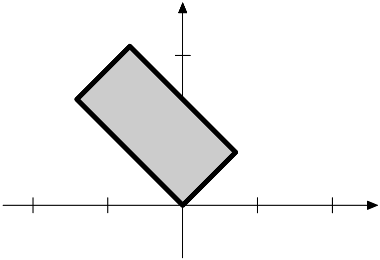
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



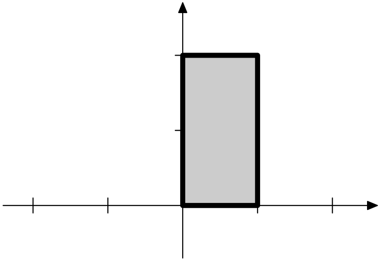
Rotation by 90°



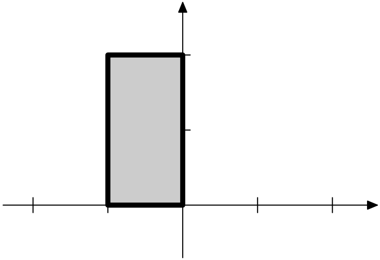
$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$



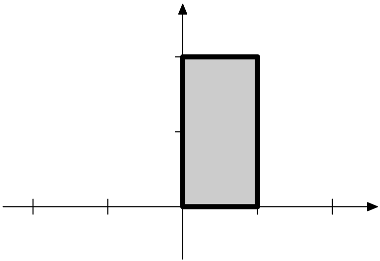
Rotation by 45^{o}



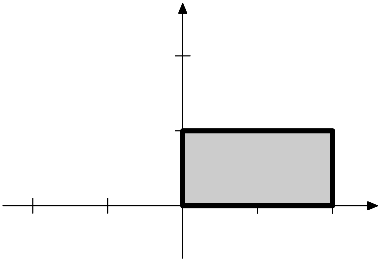
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



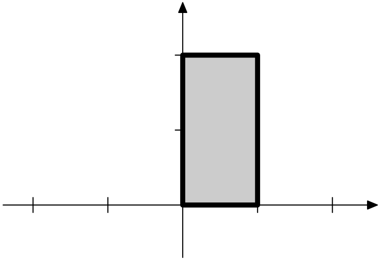
Reflection in
the vertical axis



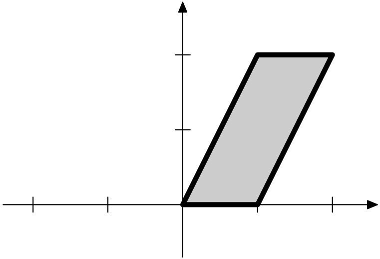
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



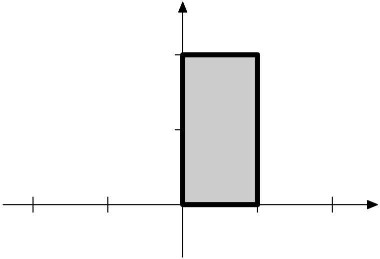
Reflection in
the line $x - y = 0$



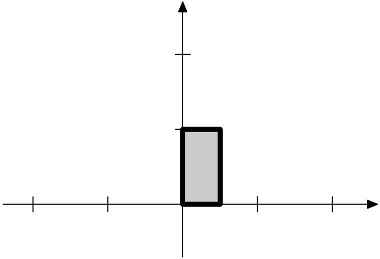
$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$



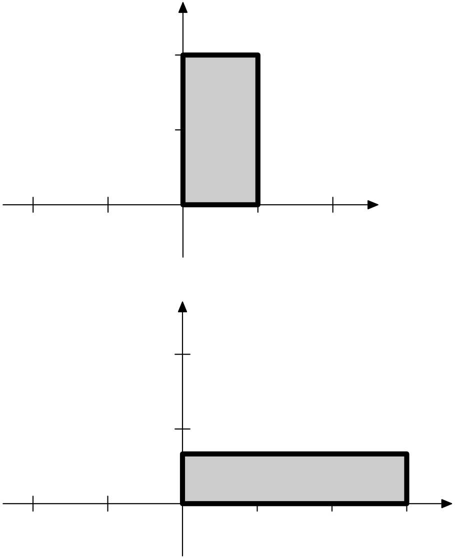
Horizontal shear



$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

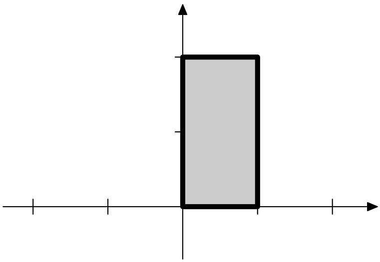


Scaling

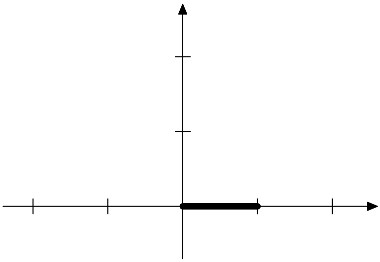


$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$$

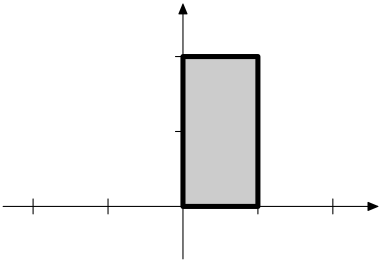
Squeeze



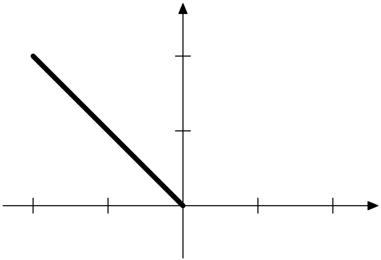
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



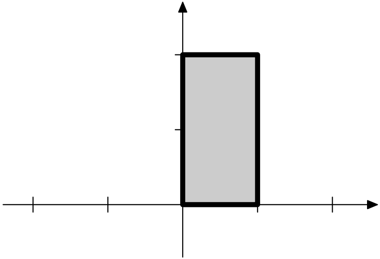
Vertical projection on the horizontal axis



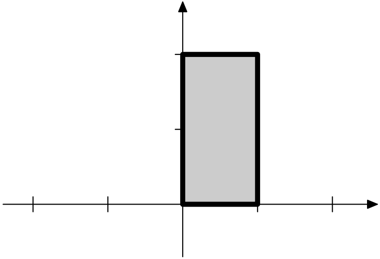
$$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$



Horizontal projection
on the line $x + y = 0$



$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Identity