MATH 311-504

Topics in Applied Mathematics

Lecture 2-7:
Basis and coordinates.

Isomorphism

Definition. A linear mapping $f : V_1 \to V_2$ is called an **isomorphism** of vector spaces if it is both one-to-one and onto.

Two vector spaces V_1 and V_2 are called **isomorphic** if there exists an isomorphism $f : V_1 \to V_2$.

The word "isomorphism" applies when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure, where "corresponding" means that the two parts play similar roles in their respective structures.

Examples of isomorphisms

- $\mathcal{M}_{2,2}(\mathbb{R})$ is isomorphic to $\mathbb{R}^4$.

Isomorphism: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)$.

- $\mathcal{M}_{2,3}(\mathbb{R})$ is isomorphic to $\mathcal{M}_{3,2}(\mathbb{R})$.

Isomorphism: $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$.

- The plane $z = 0$ in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.

Isomorphism: $(x, y, 0) \mapsto (x, y)$.

- $\mathcal{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$.

Isomorphism: $a_0 + a_1 x + \cdots + a_n x^n \mapsto (a_0, a_1, \ldots, a_n)$.

Classification problems of linear algebra

Problem 1 Given vector spaces V_1 and V_2, determine whether they are isomorphic or not.

Problem 2 Given a vector space V, determine whether V is isomorphic to $\mathbb{R}^n$ for some $n \geq 1$.

Problem 3 Show that vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not isomorphic if $m \neq n$.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

An infinite set $S \subset V$ is **linearly dependent** if there are some linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$. Otherwise S is **linearly independent**.

Theorem Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ are linearly dependent if and only if one of them is a linear combination of the other $k-1$ vectors.

Examples of linear independence:

- Vectors $\mathbf{e}_1 = (1, 0, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$,..., $\mathbf{e}_n = (0, 0, \dots, 0, 1)$ in $\mathbb{R}^n$.

- Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

- Polynomials $1, x, x^2, \dots, x^n, \dots$.

Problem. Show that functions x, e^x, and e^{-x} are linearly independent in $C(\mathbb{R})$.

Suppose that $ax + be^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Divide both sides of the identity by e^x:

$$axe^{-x} + b + ce^{-2x} = 0.$$

The left-hand side approaches b as $x \to +\infty$. $\implies b = 0$

Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the identity by x:

$$a + cx^{-1}e^{-x} = 0.$$

The left-hand side approaches a as $x \to +\infty$. $\implies a = 0$

Now $ce^{-x} = 0 \implies c = 0$.

Theorem 1 Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct real numbers. Then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}$ are linearly independent.

Theorem 2 The set of functions

$$\{x^m e^{\lambda x} \mid \lambda \in \mathbb{R},\ m = 0, 1, 2, \ldots\}$$

is linearly independent.

Spanning sets and linear dependence

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors from a vector space V.

Proposition If $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then

$$\mathrm{Span}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k =$$
$$= (t_0r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0r_k + t_k)\mathbf{v}_k.$$

Corollary Any spanning set for a vector space is minimal if and only if it is linearly independent.

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Suppose that a set $S \subset V$ is a basis for V.

"Spanning set" means that any vector $\mathbf{v} \in V$ can be represented as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \ldots, r_k \in \mathbb{R}$. "Linearly independent" implies that the above representation is unique:

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = r_1'\mathbf{v}_1 + r_2'\mathbf{v}_2 + \cdots + r_k'\mathbf{v}_k$$
$$\implies (r_1 - r_1')\mathbf{v}_1 + (r_2 - r_2')\mathbf{v}_2 + \cdots + (r_k - r_k')\mathbf{v}_k = \mathbf{0}$$
$$\implies r_1 - r_1' = r_2 - r_2' = \ldots = r_n - r_n' = 0$$

Examples. • Standard basis for $\mathbb{R}^n$:
$\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0)$, … ,
$\mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$.
Indeed, $(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

• Polynomials $1, x, x^2, \dots, x^n$ form a basis for $\mathcal{P}_n = \{a_0 + a_1x + \dots + a_nx^n : a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

Problem Let $\mathbf{v}_1 = (2, 5)$ and $\mathbf{v}_2 = (1, 3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$.

Linear independence is obvious: $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel.

To show spanning, it is enough to represent vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{e}_1 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 1 \\ 5r_1 + 3r_2 = 0 \end{cases} \iff \begin{cases} r_1 = 3 \\ r_2 = -5 \end{cases}$$

$$\mathbf{e}_2 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 0 \\ 5r_1 + 3r_2 = 1 \end{cases} \iff \begin{cases} r_1 = -1 \\ r_2 = 2 \end{cases}$$

Thus $\mathbf{e}_1 = 3\mathbf{v}_1 - 5\mathbf{v}_2$ and $\mathbf{e}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2$.

Then $(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 = x(3\mathbf{v}_1 - 5\mathbf{v}_2) + y(-\mathbf{v}_1 + 2\mathbf{v}_2)$
$= (3x - y)\mathbf{v}_1 + (-5x + 2y)\mathbf{v}_2$.

Let W be the set of all solutions of the ODE $y''(x) - y(x) = 0$. W is a subspace of the vector space $C^\infty(\mathbb{R})$ since it is the null-space of the linear operator $L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $L(f) = f'' - f$.

W contains functions e^x, e^{-x},
hyperbolic sine $\sinh x = \frac{1}{2}(e^x - e^{-x})$, and
hyperbolic cosine $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

We have that $(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$, $\cosh^2 x - \sinh^2 x = 1$.

Proposition $\{e^x, e^{-x}\}$ and $\{\cosh x, \sinh x\}$ are two bases for W.

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Proof: "Linear independence": e^x and e^{-x} are linearly independent as shown earlier.

Further, $\cosh 0 = 1$, $\sinh 0 = 0$, $\cosh' 0 = 0$, $\sinh' 0 = 1$. It follows that $\cosh x$ and $\sinh x$ are not scalar multiples of each other.

"Spanning": Take any function $f \in W$. Consider a function $g(x) = a \cosh x + b \sinh x$, where $a = f(0)$, $b = f'(0)$. We have $g(0) = a$, $g'(0) = b$.

The initial value problem $y'' - y = 0$, $y(0) = a$, $y'(0) = b$ has a unique solution. Therefore $f = g$.

Thus $f(x) = a \cosh x + b \sinh x$

$= \frac{a}{2}(e^x + e^{-x}) + \frac{b}{2}(e^x - e^{-x}) = \frac{1}{2}(a+b)e^x + \frac{1}{2}(a-b)e^{-x}$.

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \mapsto \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

is a one-to-one correspondence between V and $\mathbb{R}^n$. This correspondence is linear (hence it is an isomorphism of V onto $\mathbb{R}^n$).